IYGB - FP1 PAPER M - QUESTION 1

- $\underline{M} = \begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix}$

- $\det(\underline{M}) = k(k+2) - (k+1)(k+1) = k^2+2k-(k^2+2k+1)$

$= k^2+2k-k^2-2k-1 = -1$

- $\underline{M}^{-1} = \frac{1}{-1}\begin{bmatrix} k+2 & -(k+1) \\ -(k+1) & k \end{bmatrix} = -\begin{bmatrix} k+2 & -k-1 \\ -k-1 & k \end{bmatrix} = \begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix}$

- NOW VERIFYING BY MULTIPLICATION

$$\underline{M}\,\underline{M}^{-1} = \begin{bmatrix} k & k+1 \\ k+1 & k+2 \end{bmatrix}\begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix}$$

$$= \begin{bmatrix} k(-k-2)+(k+1)^2 & k(k+1)-k(k+1) \\ (k+1)(-k-2)+(k+1)(k+2) & (k+1)^2-k(k+2) \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{-k^2}\cancel{-2k}+\cancel{k^2}\cancel{+2k}+1 & \cancel{k^2}\cancel{+k}\cancel{-k^2}\cancel{-k} \\ \cancel{-k^2}\cancel{-3k}\cancel{-2}\cancel{+k^2}\cancel{+3k}\cancel{+2} & \cancel{k^2}\cancel{+2k}+1\cancel{-k^2}\cancel{-2k} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \underline{I}$$

- INDEED THE INVERSE

IYGB, FP1 PAPER M, QUESTION 2

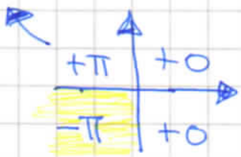
METHOD A

$$w = \frac{-9+3i}{1-2i} = \frac{(-9+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{-9-18i+3i-6}{1+\cancel{2i}-\cancel{2i}+4}$$

$$= \frac{-15-15i}{5} = -3-3i$$

- $|w| = |-3-3i| = \sqrt{(-3)^2+(-3)^2} = \sqrt{18} = 3\sqrt{2}$
- $\arg w = \arg(-3-3i) = \arctan\left(\frac{-3}{-3}\right) - \pi$

$$= \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$



METHOD B

- $|w| = \left|\frac{-9+3i}{1-2i}\right| = \frac{|-9+3i|}{|1-2i|} = \frac{\sqrt{81+9}}{\sqrt{1+4}} = \frac{\sqrt{90}}{\sqrt{5}}$

$$= \frac{\cancel{\sqrt{5}}\sqrt{2}\times\sqrt{9}}{\cancel{\sqrt{5}}} = 3\sqrt{2}$$

- $\arg w = \arg\left[\frac{-9+3i}{1-2i}\right] = \arg(-9+3i) - \arg(1-2i)$

$$= \left[\arctan\left(\frac{3}{-9}\right)+\pi\right] - \left[\arctan\left(\frac{-2}{1}\right)\right]$$ (SEE ABOVE DIAGRAM)

$$= \pi - \arctan\frac{1}{3} + \arctan 2$$

$$= \frac{5}{4}\pi$$

-2π TO GET IN RANGE

$$= -\frac{3\pi}{4}$$

IYGB - FP1 PAPER M - QUESTION 3

METHOD A - USING STANDARD ROOTS RELATIONSHIPS

$2x^2 - 8x + 9 = 0$

- $\alpha + \beta = -\frac{b}{a} = -\frac{-8}{2} = 4$
- $\alpha\beta = \frac{c}{a} = \frac{9}{2}$

PROCEED AS FOLLOWS

$A = \alpha^2 - 1$ & $B = \beta^2 - 1$

- $A + B = (\alpha^2 - 1) + (\beta^2 - 1) = \alpha^2 + \beta^2 - 2$
  $= (\alpha+\beta)^2 - 2\alpha\beta - 2 = 4^2 - 2 \times \frac{9}{2} - 2 = 5$

- $AB = (\alpha^2-1)(\beta^2-1) = \alpha^2\beta^2 - \alpha^2 - \beta^2 + 1 = (\alpha\beta)^2 - (\alpha^2+\beta^2) + 1$
  $= (\alpha\beta)^2 - \left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 1 = \left(\frac{9}{2}\right)^2 - \left[4^2 - 2 \times \frac{9}{2}\right] + 1$
  $= \frac{81}{4} - 7 + 1 = \frac{57}{4}$

HENCE THE REQUIRED QUADRATIC WILL BE

$\Longrightarrow x^2 - (A+B)x + (AB) = 0$

$\Longrightarrow x^2 - 5x + \frac{57}{4} = 0$

$\Longrightarrow 4x^2 - 20x + 57 = 0$ //

METHOD B - BY "FORCING" A SOLUTION

LET $y = x^2 - 1 \Longrightarrow x^2 = y + 1$

$\Longrightarrow x = \pm\sqrt{y+1}$

IYGB - FP1 PAPER M - QUESTION 3

SUBSTITUTE INTO THE QUADRATIC IN x

$$\Rightarrow 2\left(\pm\sqrt{y+1}\right)^2 - 8\left(\pm\sqrt{y+1}\right) + 9 = 0$$

$$\Rightarrow 2(y+1) \pm 8\sqrt{y+1} + 9 = 0$$

$$\Rightarrow \pm 8\sqrt{y+1} = -9 - 2(y+1)$$

$$\Rightarrow \pm 8\sqrt{y+1} = -9 - 2y - 2$$

$$\Rightarrow \pm 8\sqrt{y+1} = -2y - 11$$

$$\Rightarrow 64(y+1) = (-2y-11)^2$$

$$\Rightarrow 64y + 64 = 4y^2 + 44y + 121$$

$$\Rightarrow 0 = 4y^2 - 20y + 57$$

OR

$$4x^2 - 20x + 57 = 0$$

AS BEFORE

IYGB - FP1 PAPER M - QUESTION 4

START BY FINDING THE INVERSE OF $\underline{R}$ - USE ELEMENTARY ROW OPERATIONS

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad r_1(-1) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(SELF INVERSE)

PARAMETERIZE THE LINE

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{z-1}{4} = \lambda \implies \begin{array}{l} x = 3\lambda - 2 \\ y = 2\lambda + 1 \\ z = 4\lambda + 1 \end{array}$$

$$\Rightarrow \underline{X} = \underline{R}\,\underline{x}$$

$$\Rightarrow \underline{R}^{-1}\underline{X} = \underline{R}^{-1}\underline{R}\,x$$

$$\Rightarrow x = \underline{R}^{-1}\underline{X}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3\lambda-2 \\ 2\lambda+1 \\ 4\lambda+1 \end{bmatrix} = \begin{bmatrix} -3\lambda+2 \\ 2\lambda+1 \\ 4\lambda+1 \end{bmatrix}$$

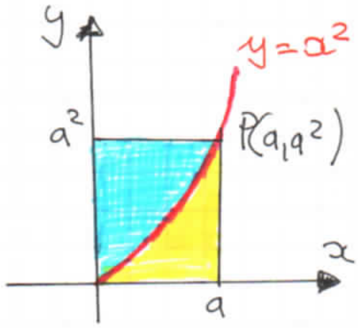
ELIMINATE λ TO GET

$$\frac{x-2}{-3} = \frac{y-1}{2} = \frac{z-1}{4} = \lambda$$

OR

$$\frac{2-x}{3} = \frac{y-1}{2} = \frac{z-1}{4}$$

IYGB - FP1 PAPER M - QUESTION 5



- LET P HAVE CO-ORDINATES (a, a^2), $a > 0$

- VOLUME OF REVOLUTION ABOUT THE x AXIS

$$V_x = \pi \int_{x_1}^{x_2} y^2 \, dx$$

$$V_x = \pi \int_0^a (x^2)^2 \, dx = \pi \int_0^a x^4 \, dx = \frac{1}{5}\pi \left[x^5 \right]_0^a = \frac{1}{5}\pi a^5$$

- VOLUME OF REVOLUTION ABOUT THE y AXIS

$$V_y = \pi \int_{y_1}^{y_2} x^2 \, dy$$

$$V_y = \pi \int_0^{a^2} y \, dy = \frac{1}{2}\pi \left[y^2 \right]_0^{a^2} = \frac{1}{2}\pi a^4$$

- NOW $V_x = V_y$

$$\frac{1}{5}\cancel{\pi} a^5 = \frac{1}{2}\cancel{\pi} a^4$$

$$a^5 = \frac{5}{2}a^4$$

$$a = \frac{5}{2}$$

$(a \neq 0)$

IYGB - FP1 PAPER M - QUESTION 6

WRITE THE COMPLEX NUMBERS IN CARTESIAN FORM

$$\boxed{z = x+iy \qquad w = u+iv \qquad \bar{w} = u-iv}$$

HENCE WE HAVE

$$|z+w|^2 - |z-\bar{w}|^2$$
$$= |x+iy+u+iv|^2 - |x+iy-(u-iv)|^2$$
$$= |(x+u)+i(y+v)|^2 - |(x-u)+i(y+v)|^2$$
$$= \left[\sqrt{(x+u)^2+(y+v)^2}\right]^2 - \left[\sqrt{(x-u)^2+(y+v)^2}\right]^2$$
$$= (x+u)^2 + \cancel{(y+v)^2} - (x-u)^2 - \cancel{(y+v)^2}$$
$$= (x+u)^2 - (x-u)^2$$
$$= (x+u+x-u)(x+u-x+u)$$
$$= (2x)(2u)$$
$$= 4xu$$
$$= 4\,\mathrm{Re}\,z\ \mathrm{Re}\,w$$

// AS REQUIRED

ALTERNATIVE METHOD USING $z\bar{z} = |z|^2$

$$|z+w|^2 - |z-\bar{w}|^2 = [z+w]\overline{[z+w]} - [z-\bar{w}]\overline{[z-\bar{w}]}$$
$$= [z+w][\bar{z}+\bar{w}] - [z-\bar{w}][\bar{z}-\bar{\bar{w}}]$$
$$= (z+w)(\bar{z}+\bar{w}) - (z-\bar{w})(\bar{z}-w)$$
$$= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} - (z\bar{z} - zw - \bar{w}\bar{z} + \bar{w}w)$$
$$= \cancel{z\bar{z}} + z\bar{w} + w\bar{z} + \cancel{w\bar{w}} - \cancel{z\bar{z}} + zw + \bar{w}\bar{z} - \cancel{\bar{w}w}$$

1~~X~~GB - FP1 PAPER M - QUESTION 6

$$= z\bar{w} + w\bar{z} + zw + \bar{w}\bar{z}$$

$$= zw + z\bar{w} + w\bar{z} + \bar{w}\bar{z}$$

$$= z(w+\bar{w}) + \bar{z}(w+\bar{w})$$

$$= (w+\bar{w})(z+\bar{z})$$

$$= (2\operatorname{Re}w)(2\operatorname{Re}z)$$

$$= 4\operatorname{Re}w\operatorname{Re}z$$

IYGB - FP1 PAPER M - QUESTION 7

$A(7,2,6) \quad B(9,10,4) \quad C(-3,-2,-2)$

a) START BY FINDING $\vec{AB}$ & $\vec{AC}$

$\vec{AB} = \overset{b}{(9,10,4)} - \overset{a}{(7,2,6)} = (2,8,-2) \sim (1,4,-1)$

$\vec{AC} = \overset{c}{(-3,-2,-2)} - \overset{a}{(7,2,6)} = (-10,-4,-8) \sim (5,2,4)$

LET THE REQUIRED VECTOR BE (p,q,r)

$$\left\{\begin{array}{l}(p,q,r)\cdot(1,4,-1)=0\\(p,q,r)\cdot(5,2,4)=0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}p+4q-r=0\\5p+2q+4r=0\end{array}\right\}$$

LET $r=1$ IN THE ABOVE EQUATIONS

$$\left\{\begin{array}{ll}p+4q-1=0 & \times -5\\5p+2q+4=0 & \times 1\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l}-5p-20q+5=0\\5p+2q+4=0\end{array}\right\} \Rightarrow -18q+9=0$$

$\Rightarrow 18q = 9$

$\Rightarrow q = \frac{1}{2}$

$\Rightarrow p+4q-1=0$

$\Rightarrow p+2-1=0$

$\Rightarrow p=-1$

HENCE A PERPENDICULAR VECTOR TO BOTH $\vec{AB}$ & $\vec{AC}$ IS

$(p,q,r) = (-1,\frac{1}{2},1) \underset{\times -2}{\sim} (2,-1,-2)$

IYGB - FP1 PAPER M - QUESTION 7

HENCE AN EQUATION OF THE REQUIRED PLANE IS

$$2x - y - 2z = \text{constant}$$

USING THE POINT $A(7,2,6)$

$$(2\times7) - 2 - (2\times6) = \text{constant}$$

$$\text{constant} = 0$$

$$\therefore\ 2x - y - 2z = 0$$

b) THE REQUIRED LINE HAS DIRECTION VECTOR $(2,-1,-2)$

$$\Rightarrow \underline{r} = (\text{FIXED POINT}) + \lambda(\text{DIRECTION VECTOR})$$

$$\Rightarrow \underline{r} = (11,3,-4) + \lambda(2,-1,-2)$$

$$\Rightarrow (x,y,z) = (2\lambda+11, -\lambda+3, -2\lambda-4)$$

SOLVING SIMULTANEOUSLY WITH $2x - y - 2z = 0$

$$\Rightarrow 2(2\lambda+11) - (-\lambda+3) - 2(-2\lambda-4) = 0$$

$$\Rightarrow 4\lambda + 22 + \lambda - 3 + 4\lambda + 8 = 0$$

$$\Rightarrow 9\lambda = -27$$

$$\Rightarrow \lambda = -3$$

$2\lambda+11$ $-\lambda-3$ $-2\lambda-4$

$$\therefore\ Q(5,6,2)$$

IYGB - FP1 PAPER N - QUESTION 7

c) FINALLY TO FIND THE DISTANCE

$$\left.\begin{array}{l} P(11,3,-4) \\ Q(5,6,2) \end{array}\right\} \Rightarrow |\overrightarrow{PQ}| = |\mathbf{q} - \mathbf{p}|$$

$$= |(5,6,2)-(11,3,-4)|$$

$$= |(-6,3,6)|$$

$$= \sqrt{36+9+36}$$

$$= \sqrt{81}$$

$$= 9$$

ALTERNATIVE

SINCE $|\text{DIRECTION VECTOR}| = |2,-1,-2| = 3$

AND $\lambda = -3$, THE REQUIRED DISTANCE

WILL BE $3 \times |\lambda| = 9$

IYGB - FP1 PAPER M - QUESTION 8

$$\boxed{f(n) = 4^{n+1} + 5^{2n-1}}$$

- **THE BASE CASE, IE $n=1$**

$$f(1) = 4^2 + 5^1 = 16 + 5 = 21 \quad \text{IE DIVISIBLE BY 21}$$

- **INDUCTIVE HYPOTHESIS**

SUPPOSE THAT $f(n)$ IS DIVISIBLE BY 21 FOR $n = k \in \mathbb{N}$, IE $f(k) = 21m$ FOR SOME $m \in \mathbb{N}$

THEN $f(k+1) - f(k) = \left(4^{k+2} + 5^{2k+1}\right) - \left(4^{k+1} + 5^{2k-1}\right)$

$$f(k+1) - 21m = 4^1 \times 4^{k+1} - 4^{k+1} + 5^2 \times 5^{2k-1} - 5^{2k-1}$$

$$f(k+1) - 21m = 4 \times 4^{k+1} - 4^{k+1} + 25 \times 5^{2k-1} - 5^{2k-1}$$

$$f(k+1) - 21m = 3 \times 4^{k+1} + 24 \times 5^{2k-1}$$

BUT $f(k) = 4^{k+1} + 5^{2k+1} = 21m$

$$f(k+1) - 21m = \left[3 \times 4^{k+1} + 3 \times 5^{2k-1}\right] + 21 \times 5^{2k-1}$$

$$f(k+1) - 21m = 3 \times f(k) + 21 \times 5^{2k-1}$$

$$f(k+1) = 84m + 21 \times 5^{2k-1}$$

$$f(k+1) = 21 \times \left[4m + 5^{2k-1}\right]$$

- **CONCLUSION**

IF $f(k)$ IS DIVISIBLE BY 21 FOR $k \in \mathbb{N}$, SO IS $f(k+1)$. SINCE $f(1)$ IS DIVISIBLE BY 21 FOR ALL $n \in \mathbb{N}$

IYGB - FP1 PAPER M - QUESTION 9

a) $\sum_{r=1}^{n}(r^3-r) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r = \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)$

$= \frac{1}{4}n(n+1)\left[n(n+1) - 2\right] = \frac{1}{4}n(n+1)(n^2+n-2)$

$= \frac{1}{4}n(n+1)(n-1)(n+2)$

b) CALCULATE IN SECTIONS

$\Rightarrow \sum_{r=5}^{10}\left[r^3 - r + 6k\right] - \sum_{r=1}^{12}(r^2+k^2) = 70$

$\Rightarrow \sum_{r=5}^{10}(r^3-r) + 6k\sum_{r=5}^{10} 1 - \sum_{r=1}^{12} r^2 - k^2\sum_{r=1}^{12} 1 = 70$

$\Rightarrow \left[\sum_{r=1}^{10}(r^3-r) - \sum_{r=1}^{4}(r^3-r)\right] + 6k(\underbrace{1+1+1+\dots+}_{6}) - \sum_{r=1}^{12} r^2 - k^2(\underbrace{1+\dots+1}_{12}) = 70$

$\Rightarrow \frac{1}{4}\times 9\times 10\times 11\times 12 - \frac{1}{4}\times 3\times 4\times 5\times 6 + 6k\times 6 - \frac{1}{6}\times 12\times 13\times 25 - k^2\times 12 = 70$

$\frac{1}{6}n(n+1)(2n+1)$

$\Rightarrow 2970 - 90 + 36k - 650 - 12k^2 = 70$

$\Rightarrow 0 = 12k^2 - 36k - 2160$

$\Rightarrow k^2 - 3k - 180 = 0$

$\Rightarrow (k-15)(k+12) = 0$

$\Rightarrow k = \begin{cases} 15 \\ -12 \end{cases}$