Name:________________

AS Further Maths

Core

2017 - 2018

Practice Papers

by Topic

(Graphic) Calculator Allowed

Work through these using the solutions with care, do not just copy them!

Complex Numbers 1

1. $$f(x) = 9x^3 - 33x^2 - 55x - 25.$$

Given that $x = 5$ is a solution of the equation $f(x) = 0$, use an algebraic method to solve $f(x) = 0$ completely.

(Total 5 marks)

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2. $$f(x) = 2x^3 - 6x^2 - 7x - 4.$$

(*a*) Show that $f(4) = 0$.

(1)

(*b*) Use algebra to solve $f(x) = 0$ completely.

(4)

(Total 5 marks)

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3. The roots of the equation

$$2z^3 - 3z^2 + 8z + 5 = 0$$

are z_1, z_2 and z_3.

Given that $z_1 = 1 + 2i$, find z_2 and z_3.

(5)

(Total 5 marks)

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4. The complex numbers z_1 and z_2 are given by

$$z_1 = p + 2\text{i} \text{ and } z_2 = 1 - 2\text{i}$$

where p is an integer.

(a) Find $\frac{z_1}{z_2}$ in the form $a + b\text{i}$ where a and b are real. Give your answer in its simplest form in terms of p.

(4)

Given that $\left|\frac{z_1}{z_2}\right| = 13$,

(b) find the possible values of p.

(4)

(Total 8 marks)

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5. The complex numbers z and w are given by

$$z = 8 + 3\text{i}, \qquad w = -2\text{i}$$

Express in the form $a + b\text{i}$, where a and b are real constants,

(a) $z - w$,

(1)

(b) zw.

(2)

(Total 3 marks)

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6. Given that $z_1 = 1 - \text{i}$,

(*a*) find arg (z_1).

(2)

Given also that $z_2 = 3 + 4\text{i}$, find, in the form $a + \text{i}b$, $a, b \in \mathbb{R}$,

(*b*) $z_1 z_2$,

(2)

(*c*) $\dfrac{z_2}{z_1}$.

(3)

In part (*b*) and part (*c*) you must show all your working clearly.

(Total 7 marks)

7.

$$z = 5 - 3\text{i}, \quad w = 2 + 2\text{i}$$

Express in the form $a + b\text{i}$, where a and b are real constants,

(*a*) z^2,

(2)

(*b*) $\dfrac{z}{w}$.

(3)

(Total 5 marks)

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8. $$z_1 = -2 + \mathrm{i}$$

(*a*) Find the modulus of z_1.

(1)

(*b*) Find, in radians, the argument of z_1, giving your answer to 2 decimal places.

(2)

The solutions to the quadratic equation

$$z^2 - 10z + 28 = 0$$

are z_2 and z_3.

(*c*) Find z_2 and z_3, giving your answers in the form $p \pm \mathrm{i}\sqrt{q}$, where p and q are integers.

(3)

(*d*) Show, on an Argand diagram, the points representing your complex numbers z_1, z_2 and z_3.

(2)

(Total 8 marks)

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9. $$z = \frac{50}{3 + 4\mathrm{i}}.$$

Find, in the form a + ib where a, $b \in \mathbb{R}$,

(*a*) z,

(2)

(*b*) z^2.

(2)

Find

(*c*) $|z|$,

(2)

(*d*) $\arg z^2$, giving your answer in degrees to 1 decimal place.

(2)

(Total 8 marks)

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10. Given that 2 and 1 – 5i are roots of the equation

$$x^3 + px^2 + 30x + q = 0, \qquad p, q \in \mathbb{R}$$

(*a*) write down the third root of the equation.

(1)

(*b*) Find the value of p and the value of q.

(5)

(*c*) Show the three roots of this equation on a single Argand diagram.

(2)

Total 8 marks)

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11. Given that $x = \frac{1}{2}$ is a root of the equation

$$2x^3 - 9x^2 + kx - 13 = 0, \qquad k \in \mathbb{R}$$

find

(*a*) the value of k,

(3)

(*b*) the other 2 roots of the equation.

(4)

(Total 7 marks)

Turn over

12. (i) The complex number w is given by

$$w = \frac{p - 4\text{i}}{2 - 3\text{i}}$$

where p is a real constant.

(*a*) Express w in the form $a + b\text{i}$, where a and b are real constants.
Give your answer in its simplest form in terms of p.

(3)

Given that $\arg w = \frac{\pi}{4}$

(*b*) find the value of p.

(2)

(ii) The complex number z is given by

$$z = (1 - \lambda\,\text{i})(4 + 3\text{i})$$

where λ is a real constant.

Given that

$$|z| = 45$$

find the possible values of λ.
Give your answers as exact values in their simplest form.

(3)

(Total 8 marks)

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13. $$z_1 = 3\text{i} \text{ and } z_2 = \frac{6}{1 + \text{i}\sqrt{3}}.$$

(*a*) Express z_2 in the form $a + \text{i}b$, where a and b are real numbers.

(2)

(*b*) Find the modulus and the argument of z_2, giving the argument in radians in terms of π.

(4)

(*c*) Show the three points representing z_1, z_2 and $(z_1 + z_2)$ respectively, on a single Argand diagram.

(2)

(Total 8 marks)

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14. The complex number z is given by

$$z = \frac{p + 2\mathrm{i}}{3 + p\mathrm{i}}$$

where p is an integer.

(a) Express z in the form $a + b\mathrm{i}$ where a and b are real. Give your answer in its simplest form in terms of p.

(4)

(b) Given that $\arg(z) = \theta$, where $\tan \theta = 1$ find the possible values of p.

(5)

(Total 9 marks)

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15. $$\mathrm{f}(x) = (4x^2 + 9)(x^2 - 2x + 5)$$

(a) Find the four roots of $\mathrm{f}(x) = 0$.

(4)

(b) Show the four roots of $\mathrm{f}(x) = 0$ on a single Argand diagram.

(2)

(Total 6 marks)

TOTAL FOR PAPER: 100 MARKS

Complex Numbers 2

1. $$z = \frac{4}{1+\text{i}}.$$

Find, in the form $a + \text{i}b$ where $a, b \in \mathbb{R}$,

(*a*) z,

(2)

(*b*) z^2.

(2)

Given that z is a complex root of the quadratic equation $x^2 + px + q = 0$, where p and q are real integers,

(*c*) find the value of p and the value of q.

(3)

(Total 7 marks)

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2. $$\text{f}(x) = (4x^2 + 9)(x^2 - 6x + 34).$$

(*a*) Find the four roots of f (x) = 0.

Give your answers in the form $x = p + \text{i}q$, where p and q are real.

(5)

(*b*) Show these four roots on a single Argand diagram.

(2)

(Total 7 marks)

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3. The roots of the equation

$$z^3 - 8z^2 + 22z - 20 = 0$$

are z_1, z_2 and z_3 .

(*a*) Given that $z_1 = 3 + \mathrm{i}$, find z_2 and z_3.

(4)

(*b*) Show, on a single Argand diagram, the points representing z_1, z_2 and z_3.

(2)

(Total 6 marks)

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4. Given that 4 and 2i – 3 are roots of the equation

$$x^3 + ax^2 + bx - 52 = 0$$

where *a* and *b* are real constants,

(*a*) write down the third root of the equation,

(1)

(*b*) find the value of *a* and the value of *b*.

(5)

(Total 6 marks)

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5. Given that $z = x + \mathrm{i}y$, find the value of *x* and the value of *y* such that

$$z + 3\mathrm{i}z^* = -1 + 13\mathrm{i}$$

where z^* is the complex conjugate of *z*.

(Total 7 marks)

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6. A complex number z is given by $z = a + 2\text{i}$,

where a is a non-zero real number.

(a) Find $z^2 + 2z$ in the form $x + \text{i}y$ where x and y are real expressions in terms of a.

(4)

Given that $z^2 + 2z$ is real,

(b) find the value of a.

(1)

Using this value for a,

(c) find the values of the modulus and argument of z, giving the argument in radians, and giving your answers to 3 significant figures.

(3)

(d) Show the points P, Q and R, representing the complex numbers z, z^2 and $z^2 + 2z$ respectively, on a single Argand diagram with origin O.

(3)

(e) Describe fully the geometrical relationship between the line segments OP and QR.

(2)

(Total 13 marks)

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7. $z_1 = 2 + 3\text{i}, \quad z_2 = 3 + 2\text{i}, \quad z_3 = a + b\text{i}, \quad a, b \in \mathbb{R}$

(a) Find the exact value of $|z_1 + z_2|$.

(2)

Given that $w = \dfrac{z_1 z_3}{z_2}$,

(b) find w in terms of a and b, giving your answer in the form $x + \text{i}y, \quad x, y \in \mathbb{R}$

(4)

Given also that $w = \dfrac{17}{13} - \dfrac{7}{13}\text{i}$,

(c) find the value of a and the value of b,

(3)

(d) find arg w, giving your answer in radians to 3 decimal places.

(2)

(Total 11 marks)

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8. $$z = 2 - \text{i}\sqrt{3}.$$

(*a*) Calculate arg z, giving your answer in radians to 2 decimal places.

(2)

Use algebra to express

(*b*) $z + z^2$ in the form $a + b\text{i}\sqrt{3}$, where a and b are integers,

(3)

(*c*) $\dfrac{z+7}{z-1}$ in the form $c + d\text{i}\sqrt{3}$, where c and d are integers.

(4)

Given that $$w = \lambda - 3\text{i},$$

where λ is a real constant, and $\arg(4 - 5\text{i} + 3w) = -\dfrac{\pi}{2}$,

(*d*) find the value of λ.

(2)

(Total 11 marks)

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9. $$z = -24 - 7\text{i}$$

(*a*) Show z on an Argand diagram.

(1)

(*b*) Calculate arg z, giving your answer in radians to 2 decimal places.

(2)

It is given that $$w = a + b\text{i}, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}.$$

Given also that $|w| = 4$ and $\arg w = \dfrac{5\pi}{6}$,

(*c*) find the values of a and b,

(3)

(*d*) find the value of $|zw|$.

(3)

(Total 9 marks)

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10. The point P represents a complex number z on an Argand diagram such that

$$|z - 6\text{i}| = 2|z - 3|.$$

(*a*) Show that, as z varies, the locus of P is a circle, stating the radius and the coordinates of the centre of this circle.

(6)

The point Q represents a complex number z on an Argand diagram such that

$$\arg(z - 6) = -\frac{3\pi}{4}.$$

(*b*) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies.

(4)

(*c*) Find the complex number for which both $|z - 6\text{i}| = 2|z - 3|$ and $\arg(z - 6) = -\frac{3\pi}{4}$.

(4)

(Total 14 marks)

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11. The complex number w is given by

$$w = 10 - 5\text{i}$$

(a) Find $|w|$.

(1)

(b) Find arg w, giving your answer in radians to 2 decimal places

(2)

The complex numbers z and w satisfy the equation

$$(2 + \text{i})(z + 3\text{i}) = w$$

(c) Use algebra to find z, giving your answer in the form $a + b\text{i}$,

where a and b are real numbers.

(4)

Given that

$$\arg(\lambda + 9\text{i} + w) = \frac{\pi}{4}$$

where λ is a real constant,

(d) find the value of λ.

(2)

(Total 9 marks)

TOTAL FOR PAPER: 100 MARKS

Matrices 1

1.

$$\mathbf{M} = \begin{pmatrix} x & x-2 \\ 3x-6 & 4x-11 \end{pmatrix}$$

Given that the matrix **M** is singular, find the possible values of x.

(Total 4 marks)

2.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} 3 & 6 \\ 11 & -8 \end{pmatrix}$$

(*a*) Find $\mathbf{A}^{-1}$

(2)

The transformation represented by the matrix **B** followed by the transformation represented by the matrix **A** is equivalent to the transformation represented by the matrix **P**.

(*b*) Find **B**, giving your answer in its simplest form.

(3)

(Total 5 marks)

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3. (i) $$\mathbf{A} = \begin{pmatrix} 2k+1 & k \\ -3 & -5 \end{pmatrix}, \text{ where } k \text{ is a constant}$$

Given that $\mathbf{B} = \mathbf{A} + 3\mathbf{I}$

where **I** is the 2 × 2 identity matrix, find

(*a*) **B** in terms of *k*,

(2)

(*b*) the value of *k* for which **B** is singular.

(2)

(ii) Given that $$\mathbf{C} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & -1 & 5 \end{pmatrix}$$

and $\mathbf{E} = \mathbf{CD}$

find **E**.

(2)

(Total 6 marks)

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4. (*a*) Given that $$\mathbf{A} = \begin{pmatrix} 3 & 1 & 3 \\ 4 & 5 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix},$$

find **AB**.

(2)

(*b*) Given that $$\mathbf{C} = \begin{pmatrix} 3 & 2 \\ 8 & 6 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 5 & 2k \\ 4 & k \end{pmatrix}, \text{ where } k \text{ is a constant}$$

and $\mathbf{E} = \mathbf{C} + \mathbf{D}$,

find the value of *k* for which **E** has no inverse.

(4)

(Total 6 marks)

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5. $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$

(*a*) Find **AB**.

(3)

Given that $\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(*b*) describe fully the geometrical transformation represented by **C**,

(2)

(*c*) write down $\mathbf{C}^{100}$.

(1)

(Total 6 marks)

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6. (*a*) Given that $\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$,

(i) find $\mathbf{A}^2$,

(ii) describe fully the geometrical transformation represented by $\mathbf{A}^2$.

(4)

(*b*) Given that $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$,

describe fully the geometrical transformation represented by **B**.

(2)

(*c*) Given that $\mathbf{C} = \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix}$,

where *k* is a constant, find the value of *k* for which the matrix **C** is singular.

(3)

(Total 9 marks)

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7. (i) Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 1 \end{pmatrix}$,

(*a*) find **AB**.

(*b*) Explain why $\mathbf{AB} \neq \mathbf{BA}$.

(4)

(ii) Given that $\mathbf{C} = \begin{pmatrix} 2k & -2 \\ 3 & k \end{pmatrix}$, where k is a real number

find $\mathbf{C}^{-1}$, giving your answer in terms of k.

(3)

(Total 7 marks)

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8. The transformation U, represented by the 2×2 matrix **P**, is a rotation through 90° anticlockwise about the origin.

(*a*) Write down the matrix **P**.

(1)

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line $y = -x$.

(*b*) Write down the matrix **Q**.

(1)

Given that U followed by V is transformation T, which is represented by the matrix **R**,

(*c*) express **R** in terms of **P** and **Q**,

(1)

(*d*) find the matrix **R**,

(2)

(*e*) give a full geometrical description of T as a single transformation.

(2)

Total 7 marks)

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9. A right angled triangle T has vertices $A(1, 1)$, $B(2, 1)$ and $C(2, 4)$. When T is transformed by the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the image is T'.

(*a*) Find the coordinates of the vertices of T'.

(2)

(*b*) Describe fully the transformation represented by **P**.

(2)

The matrices $\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ represent two transformations. When T is transformed by the matrix **QR**, the image is T''.

(*c*) Find **QR**.

(2)

(*d*) Find the determinant of **QR**.

(2)

(*e*) Using your answer to part (*d*), find the area of T''.

(3)

(Total 11 marks)

10. (i)

$$\mathbf{A} = \begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix}$$

where p is a constant.

(*a*) Find, in terms of p, the matrix **AB**

(2)

Given that

$$\mathbf{AB} + 2\mathbf{A} = k\mathbf{I}$$

where k is a constant and **I** is the 2×2 identity matrix,

(*b*) find the value of p and the value of k.

(4)

(ii)

$$\mathbf{M} = \begin{pmatrix} a & -9 \\ 1 & 2 \end{pmatrix}, \text{ where } a \text{ is a real constant}$$

Triangle T has an area of 15 square units.

Triangle T is transformed to the triangle T' by the transformation represented by the matrix **M**.

Given that the area of triangle T' is 270 square units, find the possible values of a.

(5)

(Total 11 marks)

11.

$$\mathbf{A} = \begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix}, \text{ where } a \text{ and } b \text{ are constants.}$$

Given that the matrix **A** maps the point with coordinates (4, 6) onto the point with coordinates (2, −8),

(*a*) find the value of a and the value of b.

(4)

A quadrilateral R has area 30 square units.

It is transformed into another quadrilateral S by the matrix **A**.

Using your values of a and b,

(*b*) find the area of quadrilateral S.

(4)

(Total 8 marks)

12.

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(*a*) Describe fully the single geometrical transformation *U* represented by the matrix **P**.

(2)

The transformation *U* maps the point *A*, with coordinates (p, q), onto the point *B*, with coordinates ($6\sqrt{2}$, $3\sqrt{2}$).

(*b*) Find the value of p and the value of q.

(3)

The transformation *V*, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$.

(*c*) Write down the matrix **Q**.

(1)

The transformation *U* followed by the transformation *V* is the transformation *T*. The transformation *T* is represented by the matrix **R**.

(*d*) Find the matrix **R**.

(3)

(*e*) Deduce that the transformation *T* is self-inverse.

(1)

(Total 10 marks)

13.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Given that $\mathbf{M} = (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B})$,

(*a*) calculate the matrix **M**,

(6)

(*b*) find the matrix **C** such that **MC** = **A**.

(4)

(Total 10 marks)

TOTAL FOR PAPER: 100 MARKS

Matrices 2

1. (i)

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(*a*) Describe fully the single transformation represented by the matrix **A**.

(2)

The matrix **B** represents an enlargement, scale factor –2, with centre the origin.

(*b*) Write down the matrix **B**.

(1)

(ii)

$$\mathbf{M} = \begin{pmatrix} 3 & k \\ -2 & 3 \end{pmatrix},$$ where k is a positive constant.

Triangle T has an area of 16 square units.

Triangle T is transformed onto the triangle T' by the transformation represented by the matrix **M**.

Given that the area of the triangle T' is 224 square units, find the value of k.

(3)

(Total 6 marks)

2. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

The transformation represented by **B** followed by the transformation represented by **A** is equivalent to the transformation represented by **P**.

(*a*) Find the matrix **P**.

(2)

Triangle T is transformed to the triangle T'' by the transformation represented by **P**.

Given that the area of triangle T'' is 24 square units,

(*b*) find the area of triangle T.

(3)

Triangle T'' is transformed to the original triangle T by the matrix represented by **Q**.

(*c*) Find the matrix **Q**.

(2)

(Total 7 marks)

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3. $\mathbf{X} = \begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix}$, where a is a constant.

(*a*) Find the value of a for which the matrix **X** is singular.

(2)

$$\mathbf{Y} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}.$$

(*b*) Find $\mathbf{Y}^{-1}$.

(2)

The transformation represented by **Y** maps the point A onto the point B.

Given that B has coordinates $(1 - \lambda, 7\lambda - 2)$, where λ is a constant,

(*c*) find, in terms of λ, the coordinates of point A.

(4)

(Total 8 marks)

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4. (i) $$\mathbf{A} = \begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix}, \quad \text{where } k \text{ is a real constant.}$$

Given that **A** is a singular matrix, find the possible values of k.

(4)

(ii) $$\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}$$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix **B**.

The vertices of triangle T' have coordinates (0, 0), (−20, 6) and ($10c$, $6c$), where c is a positive constant.

The area of triangle T' is 135 square units.

(*a*) Find the matrix $\mathbf{B}^{-1}$.

(2)

(*b*) Find the coordinates of the vertices of the triangle T, in terms of c where necessary.

(3)

(*c*) Find the value of c.

(3)

(Total 12 marks)

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5. (i) In each of the following cases, find a 2 × 2 matrix that represents

(*a*) a reflection in the line $y = -x$,

(*b*) a rotation of 135° anticlockwise about (0, 0),

(*c*) a reflection in the line $y = -x$ followed by a rotation of 135° anticlockwise about (0, 0).

(4)

(ii) The triangle T has vertices at the points (1, k), (3, 0) and (11, 0), where k is a constant. Triangle T is transformed onto the triangle T' by the matrix

$$\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix}$$

Given that the area of triangle T' is 364 square units, find the value of k.

(6)

(Total 10 marks)

6.

$$\mathbf{A} = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}$$

and **I** is the 2 × 2 identity matrix.

(*a*) Prove that

$$\mathbf{A}^2 = 7\mathbf{A} + 2\mathbf{I}$$

(2)

(*b*) Hence show that

$$\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A} - 7\mathbf{I})$$

(2)

The transformation represented by **A** maps the point P onto the point Q.

Given that Q has coordinates $(2k + 8, -2k - 5)$, where k is a constant,

(*c*) find, in terms of k, the coordinates of P.

(4)

(Total 8 marks)

7.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}.$$

(*a*) Show that **A** is non-singular.

(2)

(*b*) Find **B** such that $\mathbf{BA}^2 = \mathbf{A}$.

(4)

(Total 6 marks)

Turn over

8. $$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(*a*) Find det **A**.

(1)

(*b*) Find $\mathbf{A}^{-1}$.

(2)

The triangle *R* is transformed to the triangle *S* by the matrix **A**.

Given that the area of triangle *S* is 72 square units,

(*c*) find the area of triangle *R*.

(2)

The triangle *S* has vertices at the points (0, 4), (8, 16) and (12, 4).

(*d*) Find the coordinates of the vertices of *R*.

(4)

(Total 9 marks)

9.

$$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}.$$

(*a*) Find det **M**.

(1)

The transformation represented by **M** maps the point $S(2a - 7, a - 1)$, where a is a constant, onto the point $S'(25, -14)$.

(*b*) Find the value of a.

(3)

The point R has coordinates (6, 0).

Given that O is the origin,

(*c*) find the area of triangle *ORS*.

(2)

Triangle *ORS* is mapped onto triangle *OR'S'* by the transformation represented by **M**.

(*d*) Find the area of triangle *OR'S'*.

(2)

Given that

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(*e*) describe fully the single geometrical transformation represented by **A**.

(2)

The transformation represented by **A** followed by the transformation represented by **B** is equivalent to the transformation represented by **M**.

(*f*) Find **B**.

(4)

(Total 14 marks)

TOTAL FOR PAPER: 100 MARKS

Proof

1. (i) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(4n^2 - 1)$$

(6)

(Total 12 marks)

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2. (i) A sequence of numbers is defined by

$$u_1 = 6, \qquad u_2 = 27$$

$$u_{n+2} = 6u_{n+1} - 9u_n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3^n(n + 1)$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\text{f}(n) = 3^{3n-2} + 2^{3n+1} \quad \text{is divisible by } 19$$

(6)

(Total 12 marks)

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3. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\text{f}(n) = 8^n - 2^n$$

is divisible by 6.

(Total 6 marks)

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4. Prove by induction, that for $n \in \mathbb{Z}^+$,

(*a*) $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$

(6)

(*b*) $f(n) = 7^{2n-1} + 5$ is divisible by 12.

(6)

(Total 12 marks)

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5. A sequence of numbers $u_1, u_2, u_3, u_4, \ldots$, is defined by

$$u_{n+1} = 4u_n + 2, \quad u_1 = 2.$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = \frac{2}{3}(4^n - 1).$$

(5)

(Total 5 marks)

---

6. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 2^{2n-1} + 3^{2n-1}$$

is divisible by 5.

(Total 6 marks)

TOTAL FOR PAPER: 53 MARKS

Series

1. Show, using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, that

$$\sum_{r=1}^{n} 3(2r-1)^2 = n(2n+1)(2n-1), \text{ for all positive integers } n.$$

(Total 5 marks)

---

2. (*a*) Using the formula for $\sum_{r=1}^{n} r^2$ write down, in terms of n only, an expression for

$$\sum_{r=1}^{3n} r^2$$

(1)

(*b*) Show that, for all integers n, where $n > 0$,

$$\sum_{r=2n+1}^{3n} r^2 = \frac{n}{6}(an^2 + bn + c)$$

where the values of the constants a, b and c are to be found.

(4)

(Total 5 marks)

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3. (*a*) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n} (r+1)(r+4) = \frac{n}{3}(n+4)(n+5)$$

for all positive integers n.

(5)

(*b*) Hence show that

$$\sum_{r=n+1}^{2n} (r+1)(r+4) = \frac{n}{3}(n+1)(an+b)$$

where a and b are integers to be found.

(3)

(Total 8 marks)

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4. (*a*) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n}(r^3 + 6r - 3) = \frac{1}{4}n^2(n + 2n + 13)$$

for all positive integers *n*.

(5)

(*b*) Hence find the exact value of

$$\sum_{r=16}^{30}(r^3 + 6r - 3).$$

(2)

(Total 7 marks)

---

5. (*a*) Use the results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$, to prove that

$$\sum_{r=1}^{n} r(r+1)(r+5) = \frac{1}{4}n(n + 1)(n + 2)(n + 7)$$

for all positive integers *n*.

(5)

(*b*) Hence, or otherwise, find the value of

$$\sum_{r=20}^{50} r(r+1)(r+5)$$

(2)

(Total 7 marks)

---

6. (*a*) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to show that

$$\sum_{r=1}^{n} r\left(r^2-3\right)=\frac{1}{4}n\left(n+1\right)\left(n+3\right)\left(n-2\right)$$

(5)

(*b*) Calculate the value of $\sum_{r=10}^{50} r\left(r^2-3\right)$.

(3)

(Total 8 marks)

---

7. (*a*) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n}(2r-1)^2=\frac{1}{3}n(4n^2-1)$$

(6)

(*b*) Hence show that

$$\sum_{r=2n+1}^{4n}(2r-1)^2=an(bn^2-1)$$

where a and b are constants to be found.

(3)

(Total 9 marks)

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8. (*a*) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} (r+2)(r+3) = \frac{1}{3}n(n^2+9n+26)$$

for all positive integers *n*.

(6)

(*b*) Hence show that

$$\sum_{r=n+1}^{3n} (r+2)(r+3) = \frac{2}{3}n(an^2+bn+c)$$

where *a*, *b* and *c* are integers to be found.

(4)

(Total 10 marks)

---

9. (*a*) Use the results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(2n+1)(2n-1)$$

for all positive integers *n*.

(6)

(*b*) Hence show that

$$\sum_{r=n+1}^{3n} (2r-1)^2 = \frac{2}{3}n(an^2+b)$$

where *a* and *b* are integers to be found.

(4)

(Total 10 marks)

---

10. (*a*) Prove by induction

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

(5)

(*b*) Using the result in part (*a*), show that

$$\sum_{r=1}^{n} (r^3 - 2) = \frac{1}{4}n(n^3 + 2n^2 + n - 8).$$

(3)

(*c*) Calculate the exact value of $\sum_{r=20}^{50} (r^3 - 2)$.

(3)

(Total 11 marks)

TOTAL FOR PAPER: 80 MARKS

Vectors

1. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1\text{: } \mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ p \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \qquad l_2\text{: } \mathbf{r} = \begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix},$$

where λ and μ are scalar parameters and p is a constant.

The lines l_1 and l_2 intersect at the point A.

(*a*) Find the coordinates of A.

(2)

(*b*) Find the value of the constant p.

(3)

(*c*) Find the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places.

(3)

The point B lies on l_2 where $\mu = 1$.

(*d*) Find the shortest distance from the point B to the line l_1, giving your answer to 3 significant figures.

(3)

(Total 11 marks)

---

2. Relative to a fixed origin O, the point A has position vector $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and the point B has position vector $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The points A and B lie on a straight line l.

(*a*) Find $\overrightarrow{AB}$.

(2)

(*b*) Find a vector equation of l.

(2)

The point C has position vector $2\mathbf{i} + p\mathbf{j} - 4\mathbf{k}$ with respect to O, where p is a constant.

Given that AC is perpendicular to l, find

(*c*) the value of p,

(4)

(*d*) the distance AC.

(2)

(Total 10 marks)

3. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$$

where λ and μ are scalar parameters.

The lines l_1 and l_2 intersect at the point X.

(*a*) Find the coordinates of the point X.

(3)

(*b*) Find the size of the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places.

(3)

The point A lies on l_1 and has position vector $\begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix}$

(*c*) Find the distance AX, giving your answer as a surd in its simplest form.

(2)

The point Y lies on l_2. Given that the vector $\overrightarrow{YA}$ is perpendicular to the line l_1

(*d*) find the distance YA, giving your answer to one decimal place.

(2)

The point B lies on l_1 where $|\overrightarrow{AX}| = 2|\overrightarrow{AB}|$.

(*e*) Find the two possible position vectors of B.

(3)

(Total 13 marks)

Turn over

4. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1\text{: } \mathbf{r} = \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \qquad l_2\text{: } \mathbf{r} = \begin{pmatrix} -5 \\ 15 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

where μ and λ are scalar parameters.

(*a*) Show that l_1 and l_2 meet and find the position vector of their point of intersection A.

(6)

(*b*) Find, to the nearest 0.1°, the acute angle between l_1 and l_2.

(3)

The point B has position vector $\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$.

(*c*) Show that B lies on l_1.

(1)

(*d*) Find the shortest distance from B to the line l_2, giving your answer to 3 significant figures.

(4)

(Total 14 marks)

5. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = (9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}) + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$l_2 : \mathbf{r} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(*a*) Given that l_1 and l_2 meet, find the position vector of their point of intersection.

(5)

(*b*) Find the acute angle between l_1 and l_2, giving your answer in degrees to 1 decimal place.

(3)

Given that the point A has position vector $4\mathbf{i} + 16\mathbf{j} - 3\mathbf{k}$ and that the point P lies on l_1 such that AP is perpendicular to l_1,

(*c*) find the exact coordinates of P.

(6)

(Total 14 marks)

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6. Relative to a fixed origin O, the point A has position vector $(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$,

the point B has position vector $(5\mathbf{i} + 2\mathbf{j} + 10\mathbf{k})$,

and the point D has position vector $(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$.

The line l passes through the points A and B.

(*a*) Find the vector $\overrightarrow{AB}$.

(2)

(*b*) Find a vector equation for the line l.

(2)

(*c*) Show that the size of the angle BAD is 109°, to the nearest degree.

(4)

The points A, B and D, together with a point C, are the vertices of the parallelogram $ABCD$, where $\overrightarrow{AB} = \overrightarrow{DC}$.

(*d*) Find the position vector of C.

(2)

(*e*) Find the area of the parallelogram $ABCD$, giving your answer to 3 significant figures.

(3)

(*f*) Find the shortest distance from the point D to the line l, giving your answer to 3 significant figures.

(2)

(Total 15 marks)

Turn over

7. Relative to a fixed origin O, the point A has position vector $\begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$

and the point B has position vector $\begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}$.

The line l_1 passes through the points A and B.

(*a*) Find the vector $\overrightarrow{AB}$.

(2)

(*b*) Hence find a vector equation for the line l_1.

(1)

The point P has position vector $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

Given that angle PBA is θ,

(*c*) show that $\cos\theta = \frac{1}{3}$.

(3)

The line l_2 passes through the point P and is parallel to the line l_1.

(*d*) Find a vector equation for the line l_2.

(2)

The points C and D both lie on the line l_2.

Given that $AB = PC = DP$ and the x coordinate of C is positive,

(*e*) find the coordinates of C and the coordinates of D.

(3)

(*f*) find the exact area of the trapezium $ABCD$, giving your answer as a simplified surd.

(4)

(Total 15 marks)

8. With respect to a fixed origin O, the line l has equation

$$\mathbf{r} = \begin{pmatrix} 13 \\ 8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \text{ where } \lambda \text{ is a scalar parameter.}$$

The point A lies on l and has coordinates $(3, -2, 6)$.

The point P has position vector $(-p\mathbf{i} + 2p\mathbf{k})$ relative to O, where p is a constant.

Given that vector $\overrightarrow{PA}$ is perpendicular to l,

(*a*) find the value of p.

(4)

Given also that B is a point on l such that <$BPA = 45°$,

(*b*) find the coordinates of the two possible positions of B.

(5)

(Total 9 marks)

TOTAL FOR PAPER: 101 MARKS