Pure 7 – Algebraic Division and the Factor Theorem

Section 1

1a) $x^2+4x+y^2+6y-17=0$

$(x+2)^2-4+(y+3)^2-9-17=0$

$(x+2)^2+(y+3)^2=30$

Centre: $(-2,-3)$

radius: $\sqrt{30}$

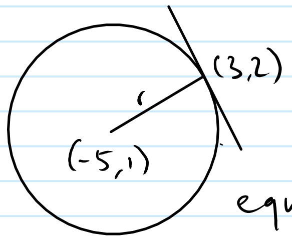
b) $x^2+y^2-6x-8y=0$

$(x-3)^2-9+(y-4)^2-16=0$

$(x-3)^2+(y-4)^2=25$

Centre: $(3,4)$

radius: 5

2a) 

gradient of r: $\frac{2-1}{3-(-5)}=\frac{1}{8}$

gradient of tangent: -8

equation of tangent: $y-2=-8(x-3)$

$y-2=-8x+24$

$y=-8x+26$

3a) $2x-y-5=0 \Rightarrow y=2x-5$ $\quad m=2 \quad \therefore m_{perp.}=-\frac{1}{2}$

$y-(-2)=-\frac{1}{2}(x-4)$

$y+2=-\frac{1}{2}x+2$

$y=-\frac{1}{2}x$

b) sub $y=-\frac{1}{2}x$ into $y=2x-5$

$-\frac{1}{2}x=2x-5$

$5=\frac{5}{2}x$

$x=2$

$y=-1$ $\quad (2,-1)$

c) $\sqrt{(4-2)^2+(-2-(-1))^2}=\sqrt{2^2+(-1)^2}$

$=\sqrt{4+1}=\sqrt{5}$

4 $5x^2-2x+1\equiv 5\left(x^2-\frac{2}{5}x+\frac{1}{5}\right)$

$\equiv 5\left[\left(x-\frac{1}{5}\right)^2-\frac{1}{25}+\frac{1}{5}\right]$

$\equiv 5\left[\left(x-\frac{1}{5}\right)^2+\frac{4}{25}\right]$

$\equiv 5\left(x-\frac{1}{5}\right)^2+\frac{4}{5}$

Min value $=\frac{4}{5}$

when $x=\frac{1}{5}$

Section 2

1. a)

$$\begin{array}{r} x-10 \\ (x+9)\overline{)x^2-x-90} \\ \underline{x^2+9x} \\ -10x-90 \\ \underline{-10x-90} \end{array}$$

b)

$$\begin{array}{r} 3x+2 \\ (x-7)\overline{)3x^2-19x-14} \\ \underline{3x^2-21x} \\ 2x-14 \\ \underline{2x-14} \end{array}$$

c)

$$\begin{array}{r} 4x-3 \\ (2x+5)\overline{)8x^2+14x-15} \\ \underline{8x^2+20x} \\ -6x-15 \\ \underline{-6x-15} \end{array}$$

d)

$$\begin{array}{r} x^2+x-1 \\ (x-1)\overline{)x^3+0x^2-2x+1} \\ \underline{x^3-x^2} \\ x^2-2x \\ \underline{x^2-x} \\ -x+1 \\ \underline{-x+1} \end{array}$$

e)

$$\begin{array}{r} x^2+x+1 \\ (x-11)\overline{)x^3-10x^2-10x-11} \\ \underline{x^3-11x^2} \\ x^2-10x \\ \underline{x^2-11x} \\ x-11 \\ \underline{x-11} \end{array}$$

f)

$$\begin{array}{r} 2x^2-7x+3 \\ (3x+4)\overline{)6x^3-13x^2-19x+12} \\ \underline{6x^3+8x^2} \\ -21x^2-19x \\ \underline{-21x^2-28x} \\ 9x+12 \\ \underline{9x+12} \end{array}$$

g)

$$\begin{array}{r} 3x^3-5x^2+4x-7 \\ (2x-3)\overline{)6x^4-19x^3+23x^2-26x+21} \\ \underline{6x^4-9x^3} \\ -10x^3+23x^2 \\ \underline{-10x^3+15x^2} \\ 8x^2-26x \\ \underline{8x^2-12x} \\ -14x+21 \\ \underline{-14x+21} \end{array}$$

h)

$$\begin{array}{r} 2x^3+7x^2-10x-1 \\ (5x-1)\overline{)10x^4+33x^3-57x^2+5x+1} \\ \underline{10x^4-2x^3} \\ 35x^3-57x^2 \\ \underline{35x^3-7x^2} \\ -50x^2+5x \\ \underline{-50x^2+10x} \\ -5x+1 \\ \underline{-5x+1} \end{array}$$

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2. a)

If $(x+6)$ is a factor of $x^3+4x^2-9x+18$, then f(–6) will be 0

$f(-6)=(-6)^3+4(-6)^2-9(-6)+18$

$=-216+144+54+18=0$

b)

If $(x-8)$ is a factor of $2x^3-13x^2-20x-32$, then f(8) will be 0

$f(8)=2(8)^3-13(8)^2-20(8)-32$

$=1024-832-160-32=0$

c)

If $(3x-1)$ is a factor of $3x^3+11x^2-25x+7$,

then $\mathrm{f}\left(\frac{1}{3}\right)$ will be 0

$$\mathrm{f}\left(\frac{1}{3}\right)=3\left(\frac{1}{3}\right)^3+11\left(\frac{1}{3}\right)^2-25\left(\frac{1}{3}\right)+7$$

$$=\frac{1}{9}+\frac{11}{9}-\frac{25}{3}+7=0$$

d)

If $(5x+2)$ is a factor of $10x^3+19x^2-39x-18$,

then $\mathrm{f}\left(\frac{-2}{5}\right)$ will be 0

$$\mathrm{f}\left(\frac{-2}{5}\right)=10\left(\frac{-2}{5}\right)^3+19\left(\frac{-2}{5}\right)^2-39\left(\frac{-2}{5}\right)-18$$

$$=\frac{-16}{25}+\frac{76}{25}+\frac{78}{5}-18=0$$

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3. a)

$x^3+3x^2-16x+12$

$\mathrm{f}(1)=(1)^3+3(1)^2-16(1)+12=0$

so $(x-1)$ is a factor.

Hence

$$\begin{array}{r} x^2+4x-12 \\ (x-1)\overline{)x^3-3x^2-16x+12} \\ \underline{x^3-x^2} \\ 4x^2-16x \\ \underline{4x^2-4x} \\ -12x+12 \\ \underline{-12x+12} \end{array}$$

However $x^2+4x-12\equiv(x+6)(x-2)$

Hence $x^3+3x^2-16x+12$

$\equiv(x+6)(x-1)(x-2)$

b)

$x^3-6x^2-55x+252$

$\mathrm{f}(1)=(1)^3-6(1)^2-55(1)+252$

$=1-6-55+252\neq0$

So $(x-1)$ is NOT a factor.

$\mathrm{f}(2)=(2)^3-6(2)^2-55(2)+252$

$=8-24-110+252\neq0$

So $(x-2)$ is NOT a factor.

$\mathrm{f}(3)=(3)^3-6(3)^2-55(3)+252$

$=27-54-165+252\neq0$

So $(x-3)$ is NOT a factor.

$\mathrm{f}(4)=(4)^3-6(4)^2-55(4)+252$

$=64-96-220+252=0$

So $(x-4)$ IS a factor.

Hence

$$\begin{array}{r} x^2-2x-63 \\ (x-4)\overline{)x^3-6x^2-55x+252} \\ \underline{x^3-4x^2} \\ -2x^2-55x \\ \underline{-2x^2+8x} \\ -63x+252 \\ \underline{-63x+252} \end{array}$$

However $x^2-2x-63\equiv(x+7)(x-9)$

Hence $x^3-6x^2-55x+252$

$\equiv(x+7)(x-9)(x-4)$

c)

$6x^3+19x^2+x-6$

$f(1)=6(1)^3+19(1)^2+(1)-6$

$=6+19+1-6\neq 0$

So $(x-1)$ is NOT a factor.

$f(-1)=6(-1)^3+19(-1)^2+(-1)-6$

$=-6+19-1-6\neq 0$

So $(x+1)$ is NOT a factor.

$f(2)=6(2)^3+19(2)^2+(2)-6$

$=48+76+2-6\neq 0$

So $(x-2)$ is NOT a factor.

$f(-2)=6(-2)^3+19(-2)^2+(-2)-6$

$=-48+76-2-6\neq 0$

So $(x+2)$ is NOT a factor.

$f(3)=6(3)^3+19(3)^2+(3)-6$

$=162+171+3-6\neq 0$

So $(x-3)$ is NOT a factor.

$f(-3)=6(-3)^3+19(-3)^2+(-3)-6$

$=-162+171-3-6=0$

So $(x+3)$ IS a factor.

Hence

$$\begin{array}{r}
6x^2+x-2 \\
(x+3)\overline{)6x^3+19x^2+x-6} \\
\underline{6x^3+18x^2} \\
x^2+x \\
\underline{x^2+3x} \\
-2x-6 \\
\underline{-2x-6}
\end{array}$$

However $6x^2+x-2\equiv(3x+2)(2x-1)$

Hence $6x^3+19x^2+x-6$
$\equiv(x+3)(2x-1)(3x+2)$

d)

x^4-13x^2-48

$\equiv(x^2+3)(x^2-16)$

$\equiv(x+4)(x-4)(x^2+3)$

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4. a)

$2(1)^4+p(1)^3-6(1)^2+q(1)+6=0$

$p+q+2=0$

b)

$2(-3)^4+p(-3)^3-6(-3)^2+q(-3)+6=0$

$-27p-3q+114=0$

$-9p-q+38=0$

c)

$(p+q+2)+(-9p-q+38)=0$

$-8p+40=0$

$p=5$

$q=-p-2=-7$

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5. a)

$f(2)=0$, so $2(2)^3+a(2)^2-4(2)+1=0$

$16+4a-8+1=0$

$4a=-9;\ a=\dfrac{-9}{4}$

b)

$f(-5)=(-5)^4+(b^2+1)(-5)^3+b(-5)^2$
$+7(-5)-15=0$

$18+b-5b^2=0$

$b=2$ or -1.8

$f(1)=(1)^4+(b^2+1)(1)^3+b(1)^2+7(1)-15$
$=0$

$b^2+b-6=0$

$b=2$ or -3

$\Rightarrow b=2$

c)

$x^2+2x-3\equiv(x-1)(x+3)$ so $f(1)$ and $f(-3)$ are both 0

Hence $2(1)^4+p(1)^3-6(1)^2+q(1)+6=0$

$p+q=-2$ and

$2(-3)^4+p(-3)^3-6(-3)^2+q(-3)+6=0$

$27p+3q=114$

Hence $27p+3(-2-p)=114$

$24p=120$

So $p=5$ and $q=-2-5=-7$

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6.

$x^2+3x-10 \equiv (x-2)(x+5)$
$x^2-9x+14 \equiv (x-2)(x-7)$

$x^3-4x^2-31x+70 \equiv (x-2)(x-7)(x+5)$

Hence the common factor is $= (x-2)$

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7.

$(5x-4)(x+3) \equiv 5x^2+11x-12$

$\equiv \frac{1}{3}(15x^2+33x-36)$

Hence $a = 15$ and $b = -36$

4

8. a)

$4x^2-12x+9 \equiv r^2$

$4x^2-12x+9 \equiv (2x-3)^2$

So $r \equiv (2x-3)$ m

b)

$(2x^3-5x^2-24x+63) \equiv \frac{1}{3}(2x+7)s^2$

Hence $6x^3-15x^2-72x+189 \equiv (2x+7)s^2$

$$\begin{array}{r|l} & 3x^2-18x+27 \\ \hline (2x+7) & 6x^3-15x^2-72x+189 \\ & \underline{6x^3+21x^2} \\ & -36x^2-72x \\ & \underline{-36x^2-126x} \\ & 54x+189 \\ & \underline{54x+189} \end{array}$$

Hence $s^2 \equiv 3x^2-18x+27 \equiv 3(x^2-6x+9)$
$\equiv 3(x-3)^2$

So the side of the square, s, is $(x-3)\sqrt{3}$ cm.

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9. a)

The body is stationary when $v = 0$,
i.e. $2t^3-19t^2+57t-54 = 0$

$f(t) = 2t^3-19t^2+57t-54$
$f(1) = 2(1)^3-19(1)^2+57(1)-54$
$= 2-19+57-54 \neq 0$ so $(x-1)$ is NOT a factor.

$f(-1) = 2(-1)^3-19(-1)^2+57(-1)-54$
$= -2-19-57-54 \neq 0$ so $(x+1)$ is NOT a factor.

$f(2) = 2(2)^3-19(2)^2+57(2)-54$
$= 16-76+114-54 = 0$ so $(x-2)$ IS a factor.

Hence $2t^3-19t^2+57t-54$
$\equiv (t-2)(At^2+Bt+C)$

Comparing coefficients of t^3 and the constant term: $A = 2$ and $-2C = -54$
so $C = 27$

Comparing coefficients of t^2:
$(B-2A) = -19$; $B-4 = -19$ so $B = -15$

Hence $2t^3-19t^2+57t-54$
$\equiv (t-2)(2t^2-15t+27)$
$\equiv (t-2)(2t-9)(t-3)$

Hence the body is stationary when $t = 2$, 3 and 4.5 s

b)

i $a = 6t^2-38t+57$
So when $t = 2$ the acceleration is
$6(2)^2-38(2)+57 = 5$

When $t = 3$ the acceleration is
$6(3)^2-38(3)+57 = -3$

And when $t = 4.5$ the acceleration is
$6(4.5)^2-38(4.5)+57 = 7.5$

ii When the acceleration is 0
$6t^2-38t+57 = 0$

$$t = \frac{38 \pm \sqrt{38^2-4\times 6\times 57}}{12} = \frac{38 \pm \sqrt{76}}{12}$$
$$= \frac{19 \pm \sqrt{19}}{6}$$

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TOTAL FOR SECTION 2:

100

Section 3

1.

$x^3+7x^2-53x-315 \equiv (x+5)(x-7)(x+9)$
and $x^3+21x^2+143x+315 \equiv (x+5)(x+7)(x+9)$

LCM $(x+5)(x-7)(x+7)(x+9)$,
HCF $(x+5)(x+9)$

2.

a) $f(x) = x^3 + (a+2)x^2 - 2x + b$

$$\left.\begin{array}{l} f(2)=0 \Rightarrow 8+4(a+2)-4+b=0 \\ f(-a)=0 \Rightarrow -a^3+(a+2)(-a)^2-2(-a)+b=0 \end{array}\right) \Rightarrow$$

$$\left.\begin{array}{l} 8+4a+8-4+b=0 \\ \cancel{-a^3}+\cancel{a^3}+2a^2+2a+b=0 \end{array}\right) \Rightarrow$$

$$\left.\begin{array}{l} b=-4a-12 \\ b=-2a^2-2a \end{array}\right) \Rightarrow -4a-12=-2a^2-2a$$

$2a^2-2a-12=0$

$a^2-a-6=0$

$(a+2)(a-3)=0$

$a = 3$ or $\cancel{-2}$

$\therefore\ b=-4\times3-12$

$b=-24$

b) $f(x) = x^3+5x^2-2x-24$

$f(x) = (x-2)(x+3)(x+4)$

$\therefore\ x = 2,\ -3,\ -4$