Pure 8 – Binomial Expansion

Section 1

1a) $2x^4 - x^3 - 5x^2 - 2x = x(2x^3 - x^2 - 5x - 2)$

$f(x) = 2x^3 - x^2 - 5x - 2$

$f(1) = 2(1)^3 - (1)^2 - 5(1) - 2 = -6 \quad \therefore\ (x-1)$ is <u>not</u> a factor of $f(x)$

$f(2) = 2(2)^3 - (2)^2 - 5(2) - 2 = 0 \quad \therefore\ (x-2)$ is a factor of $f(x)$

$(2x^3 - x^2 - 5x - 2) \div (x-2)$:

	$2x^2$	$3x$	1
x	$2x^3$	$3x^2$	x
-2	$-4x^2$	$-6x$	-2

$2x^4 - x^3 - 5x^2 - 2x = x(x-2)(2x^2 + 3x + 1)$

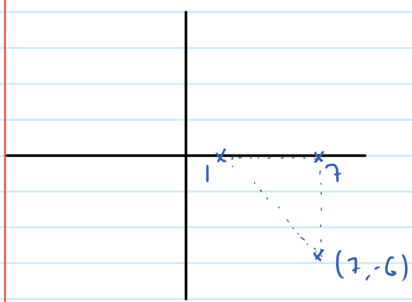
$= x(x-2)(2x+1)(x+1)$

b) $2x^4 - x^3 - 5x^2 - 2x = 0$

$x(x-2)(2x+1)(x+1) = 0$

$x = -1, -\frac{1}{2}, 0, 2$

2a)



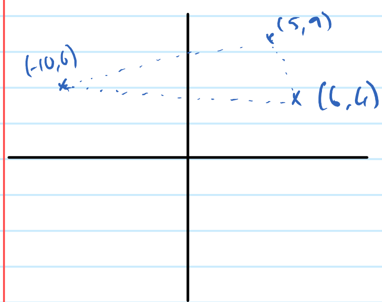
Points make a right angled triangle, so diameter goes between $(1,0)$ and $(7,-6)$

$(x-1)(x-7) + (y)(y+6) = 0$

$x^2 - 8x + 7 + y^2 + 6y = 0$

$(x-4)^2 - 16 + 7 + (y+3)^2 - 9 = 0$

$(x-4)^2 + (y+3)^2 = 18$



gradient $(5,9)$ to $(-10,6)$: $\frac{9-6}{5--10} = \frac{3}{15} = \frac{1}{5}$

gradient $(5,9)$ to $(6,4)$: $\frac{4-9}{6-5} = -5$

perpendicular gradients $\therefore$ right angled triangle
diameter is between $(-10,6)$ and $(6,4)$

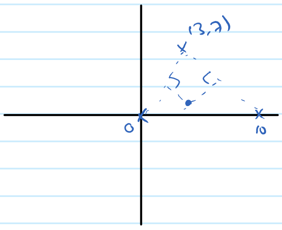
$(x+10)(x-6) + (y-6)(y-4) = 0$

$x^2 + 4x - 60 + y^2 - 10y + 24 = 0$

$(x+2)^2 - 4 - 60 + (y-5)^2 - 25 + 24 = 0$

$(x+2)^2 + (y-5)^2 = 65$

2d)



Perp. bisector of 2 chords meet at centre:

midpoint of $(0,0)$, $(3,7)$: $\left(\frac{0+3}{2}, \frac{0+7}{2}\right) = \left(\frac{3}{2}, \frac{7}{2}\right)$

gradient: $\frac{7-0}{3-0} = \frac{7}{3}$ $\therefore$ perp gradient: $-\frac{3}{7}$

$y - \frac{7}{2} = -\frac{3}{7}\left(x - \frac{3}{2}\right)$

$y - \frac{7}{2} = -\frac{3}{7}x + \frac{9}{14}$

$y = -\frac{3}{7}x + \frac{29}{7}$

midpoint of $(3,7)$, $(10,0)$: $\left(\frac{10+3}{2}, \frac{7+0}{2}\right) = \left(\frac{13}{2}, \frac{7}{2}\right)$

gradient: $\frac{0-7}{10-3} = -1$ $\therefore$ perp gradient $= 1$

$y - \frac{7}{2} = 1\left(x - \frac{13}{2}\right)$

$y = x - 3$

intercept:

$x - 3 = -\frac{3}{7}x + \frac{29}{7}$

$\frac{10}{7}x = \frac{50}{7}$

$x = 5$, $y = 2$

$(x-5)^2 + (y-2)^2 = r^2$

sub in $(0,0)$

$5^2 + 2^2 = r^2$

$29 = r^2$

$(x-5)^2 + (y-2)^2 = 29$

3a) $x^2 + 3x + k = 0$

$b^2 - 4ac > 0$

$3^2 - 4 \times 1 \times k > 0$

$9 - 4k > 0$

$k < \frac{9}{4}$

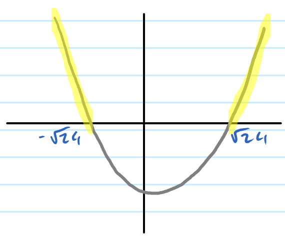
b) $3x^2 + kx + 2 = 0$

$k^2 - 4 \times 3 \times 2 > 0$

$k^2 - 24 > 0$

$(k + \sqrt{24})(k - \sqrt{24}) > 0$

$k > 2\sqrt{6}$ or $k < -2\sqrt{6}$



critical values: $\pm\sqrt{24}$

$k = \pm 2\sqrt{6}$

c) $k(x^2+1) = x - k$

$kx^2 - x + 2k = 0$

$(-1)^2 - 4 \times k \times 2k > 0$

$1 - 8k^2 > 0$

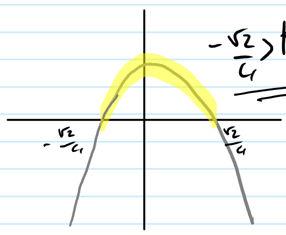
$(1 - k\sqrt{8})(1 + k\sqrt{8}) > 0$

critical values: $k = \pm\frac{1}{\sqrt{8}}$

$= \pm\frac{\sqrt{8}}{8}$

$= \pm\frac{2\sqrt{2}}{8}$

$= \pm\frac{\sqrt{2}}{4}$



$-\frac{\sqrt{2}}{4} > k > \frac{\sqrt{2}}{4}$

Section 2

1.

a) $5! = 5 \times 4 \times 3 \times 2 \times 1$
$= 120$

b) $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 5040$

c) $11! = 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 39\,916\,800$

d) $${}^5C_2 = \frac{5!}{3!2!} = \frac{5 \times 4}{2 \times 1} = 10$$

e) $${}^9C_3 = \frac{9!}{6!3!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$$

f) $${}^{11}C_7 = \frac{11!}{4!7!} = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330$$

g) $${}^{13}C_8 = \frac{13!}{5!8!} = \frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2 \times 1} = 1287$$

h) $$\binom{5}{3} = {}^5C_3 = 10$$

i) $$\binom{10}{1} = {}^{10}C_1 = 10$$

j) $$\binom{13}{5} = {}^{13}C_5 = 1287$$

k) $$\binom{20}{6} = {}^{20}C_6 = 38\,760$$

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2. a)
$$(1+x)^8 \equiv 1^8 + 1^7 \times 8x + 1^6 \times 28x^2 + 1^5 \times 56x^3 + \ldots \equiv 1 + 8x + 28x^2 + 56x^3 + \ldots$$

b)
$$(1-3x)^7 \equiv 1 + 7(-3x) + 21(-3x)^2 + 35(-3x)^3 + \ldots \equiv 1 - 21x + 189x^2 - 945x^3 + \ldots$$

c)
$$(1+2x)^9 \equiv 1 + 9(2x) + 36(2x)^2 + 84(2x)^3 + \ldots \equiv 1 + 18x + 144x^2 + 672x^3 + \ldots$$

d)
$$(2-3x)^6 \equiv 2^6 + 6(2^5)(-3x) + 15(2^4)(-3x)^2 + 20(2^3)(-3x)^3 + \ldots \equiv 64 - 576x + 2160x^2 - 4320x^3 + \ldots$$

e)
$$(x-2)^8 \equiv (-2)^8 + 8(-2)^7 x + 28(-2)^6 x^2 + 56(-2)^5 x^3 + \ldots \equiv 256 - 1024x + 1792x^2 - 1792x^3 + \ldots$$

f)
$$(2x-1)^{10} \equiv (-1)^{10} + 10(-1)^9(2x) + 45(-1)^8(2x)^2 + 120(-1)^7(2x)^3 + \ldots \equiv 1 - 20x + 180x^2 - 960x^3$$

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3. a)
$(p+5)^5$:

Term in p^2 is ${}^5C_3\, p^2 5^3$
$\equiv 1250p^2$

b)
$(4+y)^9$:

Term in y^5 is ${}^9C_5\,(4)^4 y^5$
$\equiv 32\,256y^5$

c)
$\left(z+\frac{3}{2}\right)^8$:

Term in z^6 is ${}^8C_2\,(z)^6\left(\frac{3}{2}\right)^2$
$\equiv 63z^6$

d)

i $(2a-3b)^{10}$:

Term in a^5 is ${}^{10}C_5\,(2a)^5(-3b)^5$

$\equiv -1959\,552a^5b^5$

ii $(2a-3b)^{10}$:

Term in b^4 is ${}^{10}C_4\,(2a)^6(-3b)^4$

$\equiv 1088\,640a^6b^4$

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4. a)

$$\begin{aligned}3x(2x-5)^5 &\equiv 3x((2x)^5+5(2x)^4(-5)\\&\quad+10(2x)^3(-5)^2+10(2x)^2(-5)^3\\&\quad+5(2x)(-5)^4+(-5)^5)\\&\equiv 3x(32x^5-400x^4+2000x^3\\&\quad-5000x^2+6250x-3125)\\&\equiv 96x^6-1200x^5+6000x^4\\&\quad-15\,000x^3+18\,750x^2-9375x\end{aligned}$$

b)

$$\begin{aligned}(2+x)^4(1+x) &\equiv (2^4+4(2^3)x+6(2^2)x^2\\&\quad+4(2)x^3+x^4)(1+x)\\&\equiv(16+32x+24x^2+8x^3\\&\quad+x^4)(1+x)\\&\equiv 16+32x+24x^2+8x^3+x^4\\&\quad+16x+32x^2+24x^3+8x^4+x^5\\&\equiv 16+48x+56x^2+32x^3\\&\quad+9x^4+x^5\end{aligned}$$

c)

$$\begin{aligned}&(5-2x)^3+(3+2x)^4\\&\equiv 5^3+3(5^2)(-2x)+3(5)(-2x)^2\\&\quad+(-2x)^3+3^4+4(3^3)(2x)\\&\quad+6(3^2)(2x)^2+4(3)(2x)^3+(2x)^4\\&\equiv 125-150x+60x^2-8x^3\\&\quad+81+216x+216x^2+96x^3+16x^4\\&\equiv 206+66x+276x^2+88x^3+16x^4\end{aligned}$$

d)

$$\begin{aligned}&(2+\sqrt{3})^4+(1-\sqrt{3})^4\\&=2^4+4(2^3)(\sqrt{3})+6(2^2)(\sqrt{3})^2+4(2)(\sqrt{3})^3\\&\quad+(\sqrt{3})^4+1+4(-\sqrt{3})+6(-\sqrt{3})^2\\&\quad+4(-\sqrt{3})^3+(-\sqrt{3})^4\\&=16+32\sqrt{3}+72+24\sqrt{3}+9\\&\quad+1-4\sqrt{3}+18-12\sqrt{3}+9\\&=125+40\sqrt{3}\end{aligned}$$

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5. a)

$$\begin{aligned}(1+4x)^6 &\equiv 1+6(4x)+15(4x)^2+\ldots\\&\equiv 1+24x+240x^2+\ldots\end{aligned}$$

b)

Let $x=0.01$, then $1+4x=1.04$

$(1.04)^6 \approx 1+24(0.01)+240(0.01)^2=1.264$

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6. a)

$$\begin{aligned}(1-2x)^7 &\equiv 1+7(-2x)+21(-2x)^2\\&\quad+35(-2x)^3+\ldots\\&\equiv 1-14x+84x^2-280x^3+\ldots\end{aligned}$$

b)

$1-2x=0.99 \Rightarrow x=0.005$

$$\begin{aligned}(0.99)^6 &\approx 1-14(0.005)+84(0.005)^2-280(0.005)^3\\&=0.932\,07\end{aligned}$$

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7. a)

$$\begin{aligned}1.015^5 &= 1+5(0.015)+10(0.015)^2\\&\quad+10(0.015)^3+\ldots\\&=1+0.075+0.00225\\&\quad+0.000\,03375+\ldots\\&=1.0773 \text{ to 4 decimal places.}\end{aligned}$$

b)

$$\begin{aligned}\left(\frac{199}{100}\right)^{10} &= (2-0.01)^{10}=2^{10}+10(2)^9(-0.01)\\&\quad+45(2)^8(-0.01)^2+120(2)^7(-0.01)^3\\&\quad+210(2)^6(-0.01)^4+\ldots\\&=1024-51.2+1.152\\&\quad-0.01536+\ldots\\&=973.94 \text{ to five significant figures.}\end{aligned}$$

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8. a)

$$(1+ax)^n \equiv 1 + anx + \frac{n(n-1)}{2!}(ax)^2 + \ldots$$

$$\therefore 35 = an \text{ and } \frac{n(n-1)}{2}a^2 = 490$$

$$a = \frac{35}{n} \Rightarrow \frac{n(n-1)}{2}\left(\frac{35}{n}\right)^2 = 490$$

$$\frac{1225n(n-1)}{2n^2} = 490$$

$$1225(n-1) = 980n \Rightarrow n = 5$$

b)

$$\Rightarrow a = \frac{35}{5} = 7$$

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9. a)

$$bn = -24 \Rightarrow b = -\frac{24}{n}$$

$$\frac{n(n-1)}{2}b^2 = 252$$

$$\therefore \frac{n(n-1)}{2}\left(-\frac{24}{n}\right)^2 = 252$$

$$576(n-1) = 504n \Rightarrow n = 8$$

b)

$$\Rightarrow b = -\frac{24}{8} = -3$$

6

10.

$$\frac{n(n-1)}{2}(2^2) = 8n(2)$$

$$n(n-1) = 8n$$

$$n^2 - 9n = 0$$

$$n(n-9) = 0 \Rightarrow n = 9 \text{ since } n \in \mathbb{N}$$

6

11.

$$\frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{1}{2}\right)^4$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\left(\frac{1}{2}\right)^5$$

$$\Rightarrow \frac{1}{24} = \frac{n-4}{240}$$

$$\Rightarrow n - 4 = 10$$

$$\Rightarrow n = 14$$

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Section 3

STARTING FROM THE RHS AND NOTING $\binom{N}{R} = \frac{N!}{R!\,(N-R)!}$

$$RHS = \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} + \frac{(n-1)!}{k\,[(n-1)-k]!}$$

$$= (n-1)!\left[\frac{1}{(k-1)!\,(n-k)!} + \frac{1}{k!\,(n-k-1)!}\right] = \ldots \text{ ADD FRACTIONS} \ldots$$

$$= (n-1)!\left[\frac{k}{k\,(k-1)!\,(n-k)!} + \frac{n-k}{k!\,(n-k)\,(n-k-1)!}\right]$$

$$= (n-1)!\left[\frac{k}{k!\,(n-k)!} + \frac{n-k}{k!\,(n-k)!}\right] = (n-1)!\left[\frac{n}{k!\,(n-k)!}\right]$$

$$= \frac{n\,(n-1)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k} = LHS$$

AS REQUIRED