PURE 13 SOLUTIONS – VECTORS 3

SECTION 1

1. discriminant $= 0$ $\therefore$ $(2q)^2 - 4(1)(-2q) = 0$

$\therefore$ $4q^2 + 8q = 0$

$\therefore$ $4q(q+2) = 0$

$\therefore$ $q = 0, -2$

$q \neq 0$ $\therefore$ $\underline{\underline{q = -2}}$

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2. x axis is a tangent $\therefore$ repeated root

$\therefore$ discriminant $= 0$

$\therefore$ $(r-2)^2 - 4(1)(4) = 0$

$\therefore$ $r - 2 = \pm 4$

$\therefore$ $\underline{\underline{r = 6, -2}}$

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3. $25 = 5^2$ $\therefore$ $5^{2x} = 5^{4x+1}$

$\therefore$ $2x = 4x+1$

$\therefore$ $\underline{\underline{x = -\frac{1}{2}}}$

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4. $\dfrac{10\sqrt{3}}{\sqrt{15}} \times \dfrac{\sqrt{15}}{\sqrt{15}} = \dfrac{10\sqrt{45}}{15} = \dfrac{2\sqrt{9\times 5}}{3} = 2\sqrt{5}$

$\dfrac{4}{\sqrt{5}-\sqrt{7}} \times \dfrac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}} = \dfrac{4(\sqrt{5}+\sqrt{7})}{5-7} = -2(\sqrt{5}+\sqrt{7})$

$$\therefore \frac{10\sqrt{3}}{\sqrt{15}} + \frac{4}{\sqrt{5}-\sqrt{7}} = 2\sqrt{5} - 2(\sqrt{5}+\sqrt{7})$$

$$= -2\sqrt{7} \quad \therefore \underline{\underline{k = -2}}$$

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5. $5\sqrt{3} = 2(1+\sqrt{3})^2 + p(1+\sqrt{3}) + q$

$\therefore \sqrt{3} - 8 = p + q + p\sqrt{3}$

since p and q are rational

$\sqrt{3} = p\sqrt{3} \quad \therefore \underline{\underline{p = 1}}$

$-8 = p + q \quad \therefore \underline{\underline{q = -9}}$

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6. $y = 3 - 5x - 2x^2 = -(2x^2 + 5x - 3)$

$= -2(x^2 + \frac{5}{2}x) + 3$

$x^2 + \frac{5}{2}x = (x + \frac{5}{4})^2 - \frac{25}{16}$

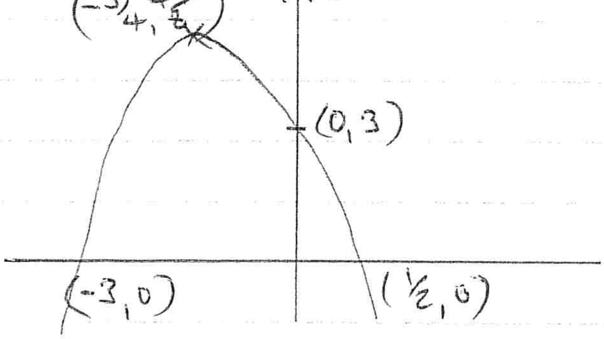
$\therefore y = -2\left((x+\frac{5}{4})^2 - \frac{25}{16}\right) + 3$

$\therefore y = -2(x+\frac{5}{4})^2 + \frac{49}{8}$

<u>turning point is $(-\frac{5}{4}, \frac{49}{8})$</u>

$y = 0 \Rightarrow 3 - 5x - 2x^2 = 0 \quad \therefore 2x^2 + 5x - 3 = 0 \quad \therefore (2x-1)(x+3) = 0$

$\therefore x = \frac{1}{2}, -3$



SECTION 2.

1 $a = 2i + j - 3k \qquad |a| = \sqrt{2^2+1^2+3^2} = \sqrt{14}$ ✓

(2)

$$\hat{a} = \frac{1}{\sqrt{14}}(2i + j - 3k)$$ ✓

2. a) $a = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

a, $a - b = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ ✓

$= \begin{pmatrix} -1 \\ -1 \\ 8 \end{pmatrix}$ ✓ (2)

b, $2a - 3b = 2\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} - 3\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ 19 \end{pmatrix}$ ✓ ✓ (2)

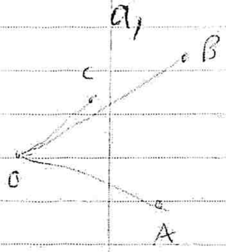
c, $a - b = \begin{pmatrix} -1 \\ -1 \\ 8 \end{pmatrix}$ is parallel to $\begin{pmatrix} 3 \\ 3 \\ -24 \end{pmatrix}$ as $\begin{pmatrix} -1 \\ -1 \\ 8 \end{pmatrix} \times -3 = \begin{pmatrix} 3 \\ 3 \\ -24 \end{pmatrix}$ ✓ ✓

(2)

$2a\,3b = \begin{pmatrix} -6 \\ -2 \\ 19 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 3 \\ 3 \\ -24 \end{pmatrix}$ as it is not a multiple of $\begin{pmatrix} -6 \\ -2 \\ 19 \end{pmatrix}$

3. $\vec{OA} = \begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} \quad \vec{AB} = \begin{pmatrix} 7 \\ -8 \\ -1 \end{pmatrix} \qquad C = (2, -2, -1)$

a,



(3)

$\vec{AB} = \vec{OB} - \vec{OA}$ ✓

$\vec{OB} = \vec{AB} + \vec{OA} = \begin{pmatrix} 7 \\ -8 \\ -1 \end{pmatrix} + \begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ ✓

$\vec{OC} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ ✓

b, $\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \\ -5 \end{pmatrix}$ ✓ ✓ (2)

c, $|\vec{AC}| = \sqrt{5^2 + 8^2 + 5^2} = \underline{\underline{\sqrt{114}}}$ ✓ Must be a surd. ✓ (2)

d, $|\vec{OC}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = \underline{\underline{3}}$ ✓ ✓ (2)

4. $a = \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}$ $b = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ $2\begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix} - \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 10 \end{pmatrix}$ ✓

$4 - p = 3$ $\therefore$ $p = 1$
$-10 - q = -6$ $\quad q = -4$ ✓✓
$12 - r = 10$ $\quad r = 2$
✓ (4)

5. $a = 3ti - 12tj + 4tk$ $\quad |a| = 39$

$\sqrt{(3t)^2 + (12t)^2 + (4t)^2} = 39$ ✓

$9t^2 + 144t^2 + 16t^2 = 1521$

$169t^2 = 1521$ ✓ (3)

$t^2 = 9$

$\underline{t = \pm 3}$ ✓

6. $\vec{AB} = -2i + 5j - 3k$ $\quad |\vec{AB}| = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38}$ ✓

<u>x axis</u> $\cos\theta = \frac{-2}{\sqrt{38}}$ ✓ $\underline{\underline{\theta = 108.9^\circ}}$ ✓

(7)

<u>y axis</u> $\cos\theta = \frac{5}{\sqrt{38}}$ ✓ $\underline{\underline{\theta = 35.8^\circ}}$ ✓

<u>z axis</u> $\cos\theta = \frac{-3}{\sqrt{38}}$ ✓ $\underline{\underline{\theta = 119.1^\circ}}$ ✓

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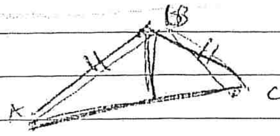

7. a) $\vec{OA} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$ $\vec{OB} = \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix}$ $\vec{OC} = \begin{pmatrix} 4 \\ 12 \\ 7 \end{pmatrix}$

$\vec{AB} = \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} 4 \\ 12 \\ 7 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}$ $\vec{BC} = \begin{pmatrix} 4 \\ 12 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix} =$

$|\vec{AB}| = \sqrt{1^2 + 5^2 + 1^2} = \sqrt{27}$ $|\vec{BC}| = \sqrt{1^2 + 5^2 + 1^2} = \sqrt{27}$

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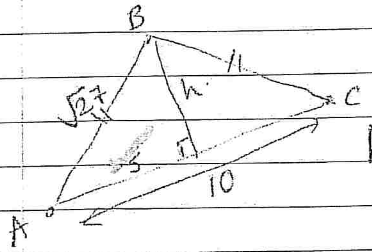
$|\vec{AC}| = \sqrt{0^2 + 10^2 + 0^2} = 10$ So $|AB| = |BC|$

and ABC is isosceles.



b) Base $= |AC| = 10$ $h = ?$

$h^2 = 27 - 5^2 = 27 - 25 = 2$

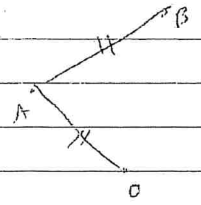
$h = \sqrt{2}$

Area $= \frac{1}{2} \times 10 \times \sqrt{2} = 5\sqrt{2}$ units^2

3

8. $\vec{OA} = \begin{pmatrix} -10 \\ 10 \\ 23 \end{pmatrix}$ $\quad \vec{OB} = \begin{pmatrix} 22 \\ p \\ -14 \end{pmatrix}$ $\quad \vec{AB} = \begin{pmatrix} 22 \\ p \\ -14 \end{pmatrix} - \begin{pmatrix} -10 \\ 10 \\ 23 \end{pmatrix} = \begin{pmatrix} 32 \\ p-10 \\ -37 \end{pmatrix}$

Option ①



$|\vec{OA}| = |\vec{AB}|$

$\left|\begin{pmatrix} -10 \\ 10 \\ 23 \end{pmatrix}\right| = \left|\begin{pmatrix} 32 \\ p-10 \\ -37 \end{pmatrix}\right|$

$\sqrt{100+100+529} = \sqrt{(p-10)^2 + 2393}$

$\sqrt{729} = \sqrt{2393 + (p-10)^2}$

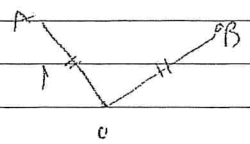
$729 = 2393 + (p-10)^2$

$-1664 = (p-10)^2$

$\sqrt{-1664}$ no solutions $\quad \therefore$ no possible value of p.

option ②

$|\vec{OA}| = |\vec{OB}|$



$\sqrt{729} = \sqrt{22^2 + p^2 + (-14)^2}$

9

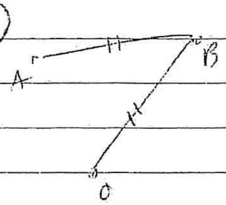
$729 = 680 + p^2$

$49 = p^2, \quad p = \pm 7$

$\therefore B = (22, 7, -14)$

or $B = (22, -7, -14)$

option ③



$|\vec{OB}| = |\vec{AB}|$

$\sqrt{680 + p^2} = \sqrt{2393 + (p-10)^2}$

$0 = -20p + 100 + 1713$

$20p = 1813$

$\therefore B = \left(22, \frac{1813}{20}, -14\right)$

$p = \frac{1813}{20}$

9.

$\vec{OP} = \frac{1}{2}c$ $\qquad$ $\vec{OQ} = \vec{OA} + \frac{1}{2}\vec{AB}$ $\qquad$ $\vec{PQ} = \vec{OQ} - \vec{OP}$

$= a + \frac{1}{2}(b-a)$ $\qquad$ $= \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c$

$= \frac{1}{2}a + \frac{1}{2}b$ $\qquad$ $= \frac{1}{2}(a+b-c)$

$\vec{OR} = \frac{1}{2}a$ $\qquad$ $\vec{OS} = \vec{OC} + \frac{1}{2}\vec{CB}$ $\qquad$ $\vec{RS} = \frac{1}{2}c + \frac{1}{2}b - \frac{1}{2}a$

$= c + \frac{1}{2}(b-c)$ $\qquad$ $= \frac{1}{2}(c+b-a)$

$= \frac{1}{2}c + \frac{1}{2}b$

$\vec{OT} = \frac{1}{2}b$ $\qquad$ $\vec{OU} = \vec{OA} + \frac{1}{2}\vec{AC}$ $\qquad$ $\vec{TU} = \frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}b$

$= a + \frac{1}{2}(c-a)$ $\qquad$ $= \frac{1}{2}(a+c-b)$

$= \frac{1}{2}a + \frac{1}{2}c$

Let X be the point of intersection of lines (vectors)

$\vec{PX} = \alpha\vec{PQ} = \alpha\frac{1}{2}(a+b-c)$

$\vec{RX} = \beta\vec{RS} = \beta\frac{1}{2}(c+b-a)$

$\vec{TX} = \lambda\vec{TU} = \lambda\frac{1}{2}(a+c-b)$

9

also $\vec{RX} = \vec{RO} + \vec{OP} + \vec{PX} = -\frac{1}{2}a + \frac{1}{2}c + \frac{\alpha}{2}(a+b-c)$

$= -\frac{1}{2}a + \frac{\alpha}{2}a + \frac{\alpha b}{2} + \frac{1}{2}c - \frac{\alpha}{2}c$

$= \left(\frac{\alpha-1}{2}\right)a + \left(\frac{\alpha}{2}\right)b + \left(\frac{1-\alpha}{2}\right)c$

$-\frac{\beta}{2}a + \frac{\beta}{2}b + \frac{\beta}{2}c =$

$(\times 2)$ $\quad -\beta a + \beta \underline{b} + \beta \underline{c} = (\alpha-1)\underline{a} + \alpha\underline{b} + (1-\alpha)\underline{c}$

Compare $\underline{a}$: $-\beta = \alpha - 1$ $\qquad$ b : $\beta = \alpha$, $\quad \beta = 1-\alpha$

$\beta = 1-\alpha \longrightarrow \beta = 1-\beta,\ 2\beta = 1,\ \beta = \frac{1}{2},\ \alpha = \frac{1}{2}$

Also $\vec{TX} = \vec{TO} + \vec{OP} + \vec{PX} = -\frac{1}{2}b + \frac{1}{2}c + \cancel{\alpha}\,\frac{1}{2}\cdot\frac{1}{2}(a+b-c)$

$\lambda\frac{1}{2}(a+c-b) = -\frac{1}{2}b + \frac{1}{2}c + \frac{1}{4}a + \frac{1}{4}b - \frac{1}{4}c$

$(\times 4)$ $2\lambda(a+c-b) = a - b + c$

Compare $\underline{a}$: $2\lambda = 1$

$\lambda = \frac{1}{2}$

So X is the midpoint of all 3 line segments PQ, RS, TU.

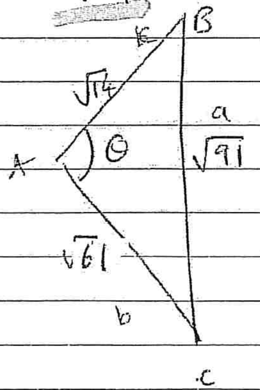
Therefore the line segments do meet at a point and bisect each other.

10. $\vec{AB} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $\vec{BC} = \begin{pmatrix} 1 \\ -9 \\ 3 \end{pmatrix}$ $\vec{AC} = \vec{BC} + \vec{AB}$

$= \begin{pmatrix} 1 \\ -9 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$

$|\vec{AB}| = \sqrt{2^2+3^2+1^2}$ $|\vec{BC}| = \sqrt{1^2+9^2+3^2}$ $|\vec{AC}| = \sqrt{3^2+6^2+4^2}$

$= \sqrt{14}$ $= \sqrt{91}$ $= \sqrt{61}$



$\cancel{\cos\theta}\ a^2 = b^2 + c^2 - 2bc\cos\theta$

$91 = 61 + 14 - 2\sqrt{61}\sqrt{14}\cos\theta$

$91 = 75 - 2\sqrt{854}\cos\theta$

$-16 = 58.446\cos\theta$

5

$\cos\theta = 0.273756$

$\theta = 105.88 = 105.9°$

TOTAL 60 MARKS