PURE 15 APPLICATIONS OF DIFFERENTIATION.

SECTION 1

1. i) $y - 3 = 4(x - 2)$ $\therefore$ $\underline{\underline{y = 4x - 5}}$

ii) $y - 2 = -\frac{1}{2}(x - 1)$ $\therefore$ $\underline{\underline{y = -\frac{1}{2}x + \frac{5}{2}}}$

iii) $y + 1 = -\frac{1}{3}(x - 4)$ $\therefore$ $\underline{\underline{y = -\frac{1}{3}x + \frac{1}{3}}}$

iv) $y = \frac{4}{3}(x + 3)$ $\therefore$ $\underline{\underline{y = \frac{4x}{3} + 4}}$

v) $y + 6 = -\frac{1}{5}(x + 1)$ $\therefore$ $\underline{\underline{y = -\frac{1}{5}x - \frac{31}{5}}}$

2. i) $(x+2)^2 - 4 + \cancel{\left(y - \frac{5}{2}\right)^2}\ y^2 - 5 = 0$

$\therefore$ $(x+2)^2 + y^2 - 9 = 0$

$\underline{\underline{(-2, 0) \quad r = 3.}}$

ii) $(x - 3)^2 + (y + 5)^2 - 9 - 25 + 20 = 0$

$(x - 3)^2 + (y + 5)^2 = 14$

$\underline{\underline{(3, -5) \quad r = \sqrt{14}}}$

iii) $(x-1)^2 + (y - 3/2)^2 - 1 - 9/4 + 3 = 0$

$(x-1)^2 + (y-3/2)^2 = 1/4$

$(1, 3/2) \qquad r = 1/2.$

3. $\text{radius}^2 = (6-4)^2 + (3 - -2)^2 = 4 + 25 = 29$

$(x-4)^2 + (y+2)^2 = 29$

4. $2y + x = 10 \qquad x^2 + y^2 = 65$

$\therefore (10 - 2y)^2 + y^2 = 65$

$\therefore 100 + 5y^2 - 40y = 65$

$\therefore 5y^2 - 40y + 35 = 0$

$\therefore y^2 - 8y + 7 = 0$

$\therefore (y-7)(y-1) = 0$

$$\left.\begin{array}{l} \therefore y = 7, 1 \\ \therefore x = -4, 8 \end{array}\right\} \quad (-4, 7) \quad (8, 1)$$

$\therefore PQ \text{ is } \sqrt{(8 - -4)^2 + (1-7)^2} = \sqrt{144 + 36}$

$\therefore PQ \text{ is } 6\sqrt{5}. \qquad = \sqrt{180}$

5. $P(-2, 6)$ $Q(6, 0)$ $R(5, 7)$

i) gradient of $PR = \frac{7-6}{5--2} = \frac{1}{7}$

gradient of $QR = \frac{7-0}{5-6} = -7$

gradient of QR = minus the reciprocal of gradient of PR $\therefore$ perpendicular.

ii) PQ subtends a ~~diamet~~ right angle $\therefore$ PQ is a diameter.

iii) length $= \sqrt{(6--2)^2 + 6^2} = 10$

$\therefore$ radius $= 5$ $\therefore$ radius$^2 = 25$.

midpoint of $PQ = (2, 3)$

hence $\underline{\underline{(x-2)^2 + (y-3)^2 = 25}}$

SECTION 2

1. i) $y = x^3 - 3x^2 + 4x - 1$

$\therefore \frac{dy}{dx} = 3x^2 - 6x + 4$

$\therefore \underline{\underline{\frac{d^2y}{dx^2} = 6x - 6}}$

ii) $y = 2x^{-1} - 3x^{-2}$

$\frac{dy}{dx} = -2x^{-2} + 6x^{-3}$ ✓

$\frac{d^2y}{dx^2} = 4x^{-3} - 18x^{-4}$ ✓

iii) $y = x^{3/2} + x^{1/2}$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2}$ ✓

$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-1/2} - \frac{1}{4}x^{-3/2}$ ✓

2. $s = t^2 + 3t - \frac{1}{t}$

$\frac{ds}{dt} = 2t + 3 + \frac{1}{t^2}$ ✓

$\frac{d^2s}{dt^2} = 2 - \frac{2}{t^3}$ ✓

$t = 2$ $\quad \frac{ds}{dt} = 2 \times 2 + 3 + \frac{1}{2^2} = 7\frac{1}{4}$ ✓

$\frac{d^2s}{dt^2} = 2 - \frac{2}{8} = 1\frac{3}{4}$ ✓

3. $y = x^4 - x + 1$

$\frac{dy}{dx} = 4x^3 - 1$

$x = 1 \therefore \frac{dy}{dx} = 4 - 1 = 3.$

$y = 1 - 1 + 1 = 1$

hence $y - 1 = 3(x - 1)$

$\therefore \underline{\underline{y = 3x - 2}}$

4. $y = x^2 - x$

$\frac{dy}{dx} = 2x - 1$

$x = 3 \quad \frac{dy}{dx} = 2 \times 3 - 1 = 5$

$\therefore$ gradient of normal $= -\frac{1}{5}$.

$y - 6 = -\frac{1}{5}(x - 3) = -\frac{1}{5}x + \frac{3}{5}$

$\therefore 5y - 30 = -x + 3$

$\therefore \underline{\underline{x + 5y = 33}}$

normal meets x-axis at $y = 0 \therefore x = 33$ ie $\underline{\underline{(33, 0)}}$

5. $y = x^3 + x + 2$

$\therefore \frac{dy}{dx} = 3x^2 + 1$

$x = 1 \quad \therefore \frac{dy}{dx} = 4 \quad$ and $\quad y = 1 + 1 + 2 = 4$

$\therefore y - 4 = 4(x - 1) = 4x - 4$

$\therefore \underline{\underline{y = 4x \quad \therefore \text{passes through origin}}}$.

gradient of normal at $P = -\frac{1}{4}$

$\therefore y - 4 = -\frac{1}{4}(x - 1) = -\frac{1}{4}x + \frac{1}{4}$

$\therefore \underline{\underline{y = -\frac{1}{4}x + \frac{17}{4}}}$

at Q, $y = 0 \quad \therefore x = 17$

OP and PQ at right-angles.

$OP^2 = \sqrt{4^2 + 1^2} = 17$

$PQ^2 = \sqrt{16^2 + 4^2} = 272$.

$\therefore$ Area of $OPQ = \frac{1}{2}\sqrt{17} \times \sqrt{272} = \sqrt{1156}$

$\therefore \underline{\underline{\text{Area} = 34}}$

6. $y = x^{-\frac{1}{2}}$ $\therefore$ $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$ ✓

$x = 1$ $\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$ ✓

$x = 1$ $\therefore$ $y = 1$ ✓

$y - 1 = -\frac{1}{2}(x - 1)$ ✓

$\therefore$ $\underline{\underline{y = -\frac{1}{2}x + \frac{3}{2}}}$ ✓

7. $y = \frac{1}{x} - \frac{2}{x^2}$ $\therefore$ $\frac{dy}{dx} = -\frac{1}{x^2} + \frac{4}{x^3}$ ✓

$x = 2$ $\therefore$ $\frac{dy}{dx} = -\frac{1}{4} + \frac{4}{8} = \frac{1}{4}$ ✓

$\therefore$ gradient of normal $= -4$ ✓

$x = 2$ $\therefore$ $y = \frac{1}{2} - \frac{2}{4} = 0$ ✓

$\therefore$ $y = -4(x - 2)$

$\therefore$ $\underline{\underline{y = -4x + 8}}$ ✓

8 i) $y = x - \frac{4}{x^2}$ $\therefore$ $\frac{dy}{dx} = 1 + \frac{8}{x^3}$ ✓

at stationary points $\frac{dy}{dx} = 0$ $\therefore$ $1 + \frac{8}{x^3} = 0$

$\therefore x = -2$

$$\frac{d^2y}{dx^2} = -\frac{24}{x^4}$$

at $x = -2$ $\quad \frac{d^2y}{dx^2} = -\frac{24}{16} < 0 \quad \therefore$ max.

<u>Stationary point at $x = -2$ (maximum)</u>

ii) $y = x^{1/2} + x^{-1/2} \quad \therefore \frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$

at $\frac{dy}{dx} = 0 \qquad x^{-1/2} - x^{-3/2} = 0$

$\therefore x - 1 = 0$

$\therefore x = 1$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2}$$

at $x = 1$ $\quad \frac{d^2y}{dx^2} = -\frac{1}{4} + \frac{3}{4} = \frac{1}{2} > 0 \quad \therefore$ min.

<u>minimum point at $x = 1$</u>

SECTION 3.

1. $y = x^3 + px^2 + q$ min. at $(4, -11)$

$\therefore \frac{dy}{dx} = 3x^2 + 2px$

at min. point $\frac{dy}{dx} = 0$ $\therefore 3x^2 + 2px = 0$

$\therefore x(3x + 2p) = 0$

at $x = 4$ $12 + 2p = 0$ $\therefore p = -6$.

$\therefore y = x^3 - 6x^2 + q$

$\therefore -11 = 64 - 6 \times 4^2 + q = -32 + q$

$\therefore q = 21$

$\frac{dy}{dx} = 0$ at $x = 0$ $\therefore y = q = 21$

$\frac{d^2y}{dx^2} = 6x + 2p = 6x - 12$

$x = 0$ $\frac{d^2y}{dx^2} = -12$ $\therefore$ max.

hence, maximum point is $(0, 21)$

2. i) $y = x^3 + ax^2 + bx + c$

$\therefore$ at $(1,1)$ $\quad 1 = 1 + a + b + c$

$$\therefore \underline{\underline{a + b + c = 0}}$$

ii) $\dfrac{dy}{dx} = 3x^2 + 2ax + b$

$\dfrac{dy}{dx} = 0$ at $x = -1$ and $x = 3$.

$\therefore 3 - 2a + b = 0$

$27 + 6a + b = 0$

$\therefore 24 + 8a = 0 \quad \therefore a = -3.$

$b = 2a - 3 = -9$

$a + b + c = 0 \quad \therefore -12 + c = 0 \quad \therefore c = 12.$

$$\underline{\underline{a = -3, \ b = -9, \ c = 12.}}$$