Solutions to Pure 16

Section 1

1 a $f'(x) = 24 - 6x - 3x^2$

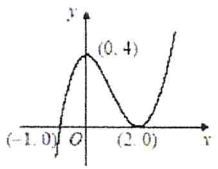
b $24 - 6x - 3x^2 \geq 0$

$x^2 + 2x - 8 \leq 0$

$(x - 2)(x + 4) \leq 0$

$-2 \leq x \leq 4$

2. a



b $f(x) = (x + 1)(x^2 - 4x + 4)$

$= x^3 - 4x^2 + 4x + x^2 - 4x + 4$

$= x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x$

c $x = 1 \therefore y = 2 \times (-1)^2 = 2$

grad $= 3 - 6 = -3$

$\therefore y - 2 = -3(x - 1)$

$y - 2 = -3x + 3$

$y = 5 - 3x$

3. a $\frac{dy}{dx} = 9 + 6x - 3x^2$

SP: $9 + 6x - 3x^2 = 0$

$-3(x + 1)(x - 3) = 0$

$x = -1, 3$

$\therefore (-1, -3)$ and $(3, 29)$

b $\frac{d^2y}{dx^2} = 6 - 6x$

$(-1, -3)$: $\frac{d^2y}{dx^2} = 12 \therefore$ minimum

$(3, 29)$: $\frac{d^2y}{dx^2} = -12 \therefore$ maximum

c $-3 < k < 29$

4. a $f(-1) = 15$

$\therefore -4 + a + 12 + b = 15$

$a + b = 7 \quad (1)$

b $f(2) = 42$

$\therefore 32 + 4a - 24 + b = 42$

$4a + b = 34 \quad (2)$

$(2) - (1) \quad 3a = 27$

$\therefore a = 9, b = -2$

c $f(x) = 4x^3 - 9x^2 - 12x - 2$

$f'(x) = 12x^2 - 18x - 12$

SP: $12x^2 - 18x - 12 = 0$

$2x^2 - 3x - 2 = 0$

$(2x + 1)(x - 2) = 0$

$x = -2, \frac{1}{2}$

$\therefore (-2, 26)$ and $(\frac{1}{2}, -\frac{21}{4})$

Section 2

1. (i) $f'(x) = x^4$

$f(x) = \frac{1}{5}x^5 + c$ ✓

(ii) $f'(x) = 2x^7$

$f(x) = 2\left(\frac{1}{8}x^8\right) + c$

$= \frac{1}{4}x^8 + c$ ✓

(iii) $f'(x) = \frac{1}{x^3} = x^{-3}$

$f(x) = \frac{1}{-2}x^{-2} + c$ ✓

$\left(= -\frac{1}{2x^2} + c\right)$

(iv) $f'(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$f(x) = \frac{3}{4}x^{\frac{4}{3}} + c$ ✓

(4)

2. (i) $\frac{dy}{dx} = 4x^2 + x$

$y = \frac{4}{3}x^3 + \frac{1}{2}x^2 + c$ ✓

When $x = 1, y = 2$

$2 = \frac{4}{3} \times 1^3 + \frac{1}{2} \times 1^2 + c$

$c = 2 - \frac{4}{3} - \frac{1}{2} = \frac{1}{6}$ ✓

$y = \frac{4}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}$ ✓

(ii) When $x = 3$, $y = \frac{4}{3} \times 3^3 + \frac{1}{2} \times 3^2 + \frac{1}{6}$

$= 36 + \frac{9}{2} + \frac{1}{6}$

$= 40\frac{2}{3}$ ✓

(4)

3. $\frac{dy}{dx} = (x-1)(3x-5) = 3x^2 - 8$

$y = 3\left(\frac{1}{3}x^3\right) - 8\left(\frac{1}{2}x^2\right) + 5x + c$

$= x^3 - 4x^2 + 5x + c$

When $x = 1, y = 2$

$2 = 1^3 - 4 \times 1^2 + 5 \times 1 + c$

$c = 2 - 1 + 4 - 5 = 0$

$y = x^3 - 4x^2 + 5x$

(5)

4. $\frac{dy}{dx} = \frac{x-3}{x^3} = x^{-2} - 3x^{-3}$

$y = -x^{-1} - 3\left(-\frac{1}{2}x^{-2}\right) + c$

$\left(= -\frac{1}{x} + \frac{3}{2x^2} + c\right)$

When $x = 1, y = 1$

$1 = -1 + \frac{3}{2} + c$

$c = \frac{1}{2}$

$y = -\frac{1}{x} + \frac{3}{2x^2} + \frac{1}{2}$

(6)

or $y = -x^{-1} + \frac{3}{2}x^{-2} + \frac{1}{2}$

5. (i) $\int_{-1}^{1} (4x+5)\,dx = \left[2x^2 + 5x\right]_{-1}^{1}$

$= 2 + 5 - (2 - 5)$

$= 7 - (-3)$

$= 10$

(3)

(ii) $\int_{-1}^{0} (6x^2 - 2x)\,dx = \left[2x^3 - x^2\right]_{-1}^{0}$

$= 0 - (-2 - 1)$

$= 3$

(3)

(iii) $\int_{2}^{4} (x^2 - x + 3)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x\right]_{2}^{4}$

$= \left(\frac{64}{3} - 8 + 12\right) - \left(\frac{8}{3} - 2 + 6\right)$

$= \frac{64}{3} + 4 - \frac{8}{3} - 4$

$= \frac{56}{3}$

(3)

(iv) $\int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2}$

$= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$

$= 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$

$= 8 - 3 - \frac{1}{2}$

$= \frac{9}{2}$

(3)

(v) $\int_{-1}^{2} (x^3 - x + 4)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 + 4x\right]_{-1}^{2}$

$= (4 - 2 + 8) - \left(\frac{1}{4} - \frac{1}{2} - 4\right)$

$= 10 + \frac{1}{4} + 4$

$= 14.25$

(3)

(vi) $\int_{1}^{3} \frac{1}{x^3}\,dx = \int_{1}^{3} x^{-3}\,dx$

$= \left[-\frac{1}{2}x^{-2}\right]_{1}^{3}$

$= \left[-\frac{1}{2x^2}\right]_{1}^{3}$

$= -\frac{1}{18} + \frac{1}{2}$

$= \frac{4}{9}$

(3)

(vii) $\int_{1}^{9} \frac{1}{\sqrt{x}}\,dx = \int_{1}^{9} x^{-\frac{1}{2}}\,dx$

$= \left[2x^{\frac{1}{2}}\right]_{1}^{9}$

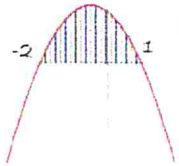
$= 6 - 2$

$= 4$

(3)

6. (i) $y = (1 - x)(x + 2)$

The graph cuts the x-axis at $x = 1$ and $x = -2$.
The coefficient of x^2 is negative, so the graph is "upside down".

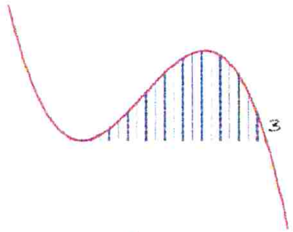


$$\text{Area} = \int_{-2}^{1} (1 - x)(x + 2)\,dx$$
$$= \int_{-2}^{1} (2 - x - x^2)\,dx$$
$$= \left[2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_{-2}^{1}$$
$$= \left(2 - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(-4 - 2 + \tfrac{8}{3}\right)$$
$$= 2 - \tfrac{1}{2} - \tfrac{1}{3} + 6 - \tfrac{8}{3}$$
$$= \tfrac{9}{2} \text{ square units}$$

(5)

(ii) $y = 3x^2 - x^3 = x^2(3 - x)$

The graph cuts the x-axis at $(3, 0)$ and touches the x-axis at the origin.



$$\text{Area} = \int_0^3$$
$$= \left[x^3 - \tfrac{1}{4}x^4\right]_0^3$$
$$= 27 - \tfrac{81}{4} - 0$$
$$= 6.75 \text{ square units}$$

(5)

(ii) $y = x(x - 1)$

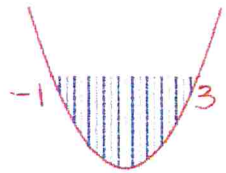
The graph cuts the x-axis at the origin and the point $(1, 0)$.

Area between $x = 0$ and $x = 1$ $= \int_0^1 x(x - 1)\,dx$
$$= \int_0^1 (x^2 - x)\,dx$$
$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^1$$
$$= \tfrac{1}{3} - \tfrac{1}{2}$$
$$= -\tfrac{1}{6}$$

Area $= \frac{1}{6}$ square units.

(5)

(iv) $y = x^2 - 2x - 3 = (x - 3)(x + 1)$



Area between $x = -1$ and $x = 3$ $= \left[\tfrac{1}{3}x^3 - x^2 - 3x\right]_{-1}^{3}$
$$= (9 - 9 - 9) - \left(-\tfrac{1}{3} - 1 + 3\right)$$
$$= -9 - \tfrac{5}{3}$$
$$= -\tfrac{32}{3}$$

Area $= \frac{32}{3}$ square units

(5)

Section 3

1. $\int_{1}^{4} (3x^2 + ax - 5)\ dx = [x^3 + \frac{1}{2}ax^2 - 5x]_1^4$

   $= (64 + 8a - 20) - (1 + \frac{1}{2}a - 5) = 48 + \frac{15}{2}a$

   $\therefore\ 48 + \frac{15}{2}a = 18$

   $a = -4$

2. $\int_{-1}^{k} (3x^2 - 12x + 9)\ dx = [x^3 - 6x^2 + 9x]_{-1}^{k}$

   $= (k^3 - 6k^2 + 9k) - (-1 - 6 - 9) = k^3 - 6k^2 + 9k + 16$

   $\therefore\ k^3 - 6k^2 + 9k + 16 = 16$

   $k(k^2 - 6k + 9) = 0$

   $k(k - 3)^2 = 0$

   $k \neq 0\ \therefore k = 3$

3. a $x^3 - 5x^2 + 6x = 0$

   $x(x - 2)(x - 3) = 0$

   $x = 0, 2, 3$

   $\therefore$ (0, 0), (2, 0) and (3, 0)

   b $\int_{0}^{2} (x^3 - 5x^2 + 6x)\ dx$

   $= [\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2]_0^2$

   $= (4 - \frac{40}{3} + 12) - 0 = \frac{8}{3}$

   $\int_{2}^{3} (x^3 - 5x^2 + 6x)\ dx$

   $= [\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2]_2^3$

   $= (\frac{81}{4} - 45 + 27) - \frac{8}{3} = -\frac{5}{12}$

   total area $= \frac{8}{3} + \frac{5}{12} = 3\frac{1}{12}$