PURE 17 FURTHER INTEGRATION.

SECTION 1

1. $f(x) = 2x^3 + 5x^2 - 1$

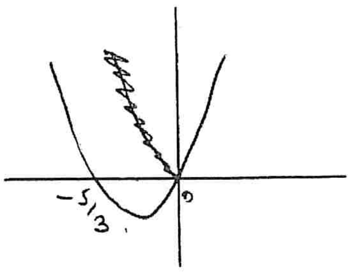
a) $f'(x) = 6x^2 + 10x$

b) $f(x)$ is increasing for $f'(x) \geq 0$

$\therefore 6x^2 + 10x \geq 0$

$\therefore 2x(3x+5) \geq 0$

$\therefore$ critical values are $x = 0, -5/3$.



ie $f'(x) \geq 0$ for $x \leq -5/3, \ x \geq 0$.

$\therefore$ $\underline{f(x) \text{ is increasing for } x \leq -5/3, \ x \geq 0}$.

2. $y = x^3 - x^2 + 2x - 4$

a) $\dfrac{dy}{dx} = 3x^2 - 2x + 2 \quad \therefore \left.\dfrac{dy}{dx}\right|_{x=1} = 3 - 2 + 2 = 3.$

$\therefore$ equation of tangent is $y + 2 = 3(x-1) = 3x - 3$

$\therefore$ $\underline{y = 3x - 5}$.

b) stationary points are where $\dfrac{dy}{dx} = 0 \ \therefore 3x^2 - 2x + 2 = 0$

Discriminant $= b^2 - 4ac$ with $b = -2$ $a = 3$ $c = 2$

$\therefore b^2 - 4ac = (-2)^2 - 4 \times 3 \times 2 = 4 - 24 = -20.$

ie negative discriminant $\therefore$ no roots, ie $3x^2 - 2x + 2$ cannot be zero $\therefore \frac{dy}{dx} \neq 0 \therefore$ no stationary points.

3 a) $\int_1^3 2 - \frac{1}{x^2}\, dx = \int_1^3 2 - x^{-2}\, dx = \left[2x + x^{-1}\right]_1^3$

$= (6 + \tfrac{1}{3}) - (2 + 1)$

$= 3\tfrac{1}{3}$ or $\frac{10}{3}$.

b) $\int_{-2}^{-1} 6x + \frac{4}{x^3}\, dx = \int_{-2}^{-1} 6x + 4x^{-3}\, dx = \left[3x^2 - \frac{2}{x^2}\right]_{-2}^{-1}$

$= (3 - 2) - (12 - \tfrac{1}{2})$

$= -10\tfrac{1}{2}$ or $-\frac{21}{2}$

c) $\int_1^4 3x^{\frac{1}{2}} - 4\, dx = \left[2x^{3/2} - 4x\right]_1^4 = (16 - 16) - (2 - 4) = 2$

d) $\int_{-1}^{2} \frac{4x^4 - x}{2x}\, dx = \int_{-1}^{2} 2x^3 - \tfrac{1}{2}\, dx = \left[\frac{x^4}{2} - \frac{x}{2}\right]_{-1}^{2}$

$= (8 - 1) - (\tfrac{1}{2} + \tfrac{1}{2}) = 6$

e) $\int_1^8 x - x^{-1/3}\, dx = \left[\frac{x^2}{2} - \frac{3}{2} x^{2/3}\right]_1^8 = (32 - 6) - (\tfrac{1}{2} - \tfrac{3}{2})$

$= 27$

f) $\int_2^3 \frac{1-6x^3}{3x^2}\,dx = \int_2^3 \frac{x^{-2}}{3} - 2x \quad dx = \left[-\frac{1}{3x} - x^2\right]_2^3$

$= \left(-\frac{1}{9} - 9\right) - \left(-\frac{1}{6} - 4\right)$

$= \underline{\underline{-4\frac{17}{18} \text{ or } -\frac{89}{18}}}$

4. a) at A $y = 0$ $\therefore$ $4x^{\frac{1}{2}} - x^{\frac{3}{2}} = 0$

$\therefore x^{\frac{1}{2}}(4 - x) = 0$

$\therefore x = 0, 4$

$\therefore$ at A $x = 4$ $\therefore$ $\underline{\underline{A \text{ is } (4, 0)}}$

b) $y = 4x^{\frac{1}{2}} - x^{\frac{3}{2}}$ $\therefore$ $\frac{dy}{dx} = 2x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$

at B $\frac{dy}{dx} = 0$ $\therefore$ $2x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = 0$

$\therefore 2 - \frac{3}{2}x = 0$

$\therefore \underline{\underline{x = \frac{4}{3}}}$

c) Area $= \int_0^4 4x^{\frac{1}{2}} - x^{\frac{3}{2}}\,dx = \left[\frac{8x^{\frac{3}{2}}}{3} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^4$

$= \frac{64}{3} - \frac{64}{5} = \underline{\underline{8\frac{8}{15} \text{ or } \frac{128}{15}}}$

SECTION 2.

1. a) at points of intersection $x^2 - 3x + 4 = x + 1$

$\therefore x^2 - 4x + 3 = 0$

$\therefore (x-3)(x-1) = 0$

$\therefore x = 1, 3.$

$x = 1 \quad y = 1 + 1 = 2$

$x = 3 \quad y = 3 + 1 = 4$

$\therefore$ points are $(1,2)$ and $(3,4)$

b) $$\int_1^3 x^2 - 3x + 4 \, dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 4x\right]_1^3$$

$$= \left(9 - \tfrac{27}{2} + 12\right) - \left(\tfrac{1}{3} - \tfrac{3}{2} + 4\right)$$

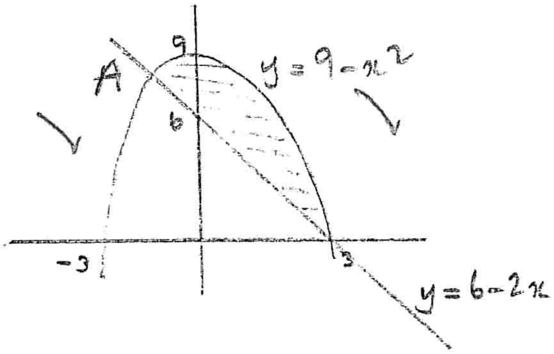
$$= \tfrac{14}{3}.$$

$$\int_1^3 x + 1 \, dx = \left[\frac{x^2}{2} + x\right]_1^3 = \left(\tfrac{9}{2} + 3\right) - \left(\tfrac{1}{2} + 1\right)$$

$$= 6$$

$\therefore$ shaded area $= 6 - \tfrac{14}{3} = \tfrac{4}{3}$

2 . a)



$9-x^2 = 6-2x$

$\therefore x^2-2x-3=0$

$\therefore (x+1)(x-3)=0$

$\therefore x=-1, 3$

$\therefore$ A is $(-1, 8)$

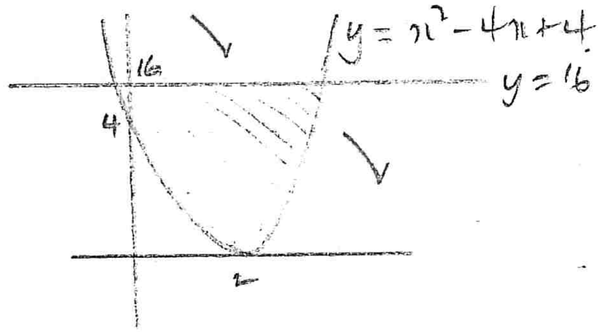
$$\int_{-1}^{3} 9-x^2 \, dx = \left[9x - \frac{x^3}{3}\right]_{-1}^{3} = (27-9)-(-9+\tfrac{1}{3})$$

$$= 26\tfrac{2}{3}$$

Area of triangle $= \frac{1}{2} \times 4 \times 8 = 16$

$\therefore$ shaded area $= 26\tfrac{2}{3} - 16 = 10\tfrac{2}{3}$

b).



$x^2-4x+4=16$

$\therefore x^2-4x-12=0$

$\therefore (x-6)(x+2)=0$

$\therefore x=6, -2$

$$\int_{-2}^{6} x^2-4x+4 \, dx$$

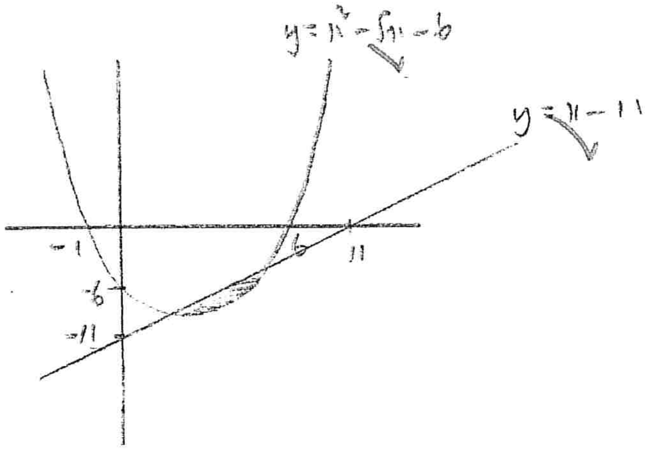
$$= \left[\frac{x^3}{3} - 2x^2 + 4x\right]_{-2}^{6} = (72-72+24)-(-\tfrac{8}{3}-8-8)$$

$$= 42\tfrac{2}{3}$$

area of rectangle $= 8 \times 16 = 128$ $\therefore$ shaded area $= 128 - 42\tfrac{2}{3}$

$= 85\tfrac{1}{3}$

c)



$x^2 - 5x - 6 = x - 11$

$\therefore x^2 - 6x + 5 = 0$

$\therefore (x-1)(x-5) = 0$

$\therefore x = 1, 5.$

$$\int_1^5 x^2 - 5x - 6 \, dx = \left[\frac{x^3}{3} - \frac{5x^2}{2} - 6x \right]_1^5$$

$$= \left(\frac{125}{3} - \frac{125}{2} - 30 \right) - \left(\frac{1}{3} - \frac{5}{2} - 6 \right)$$

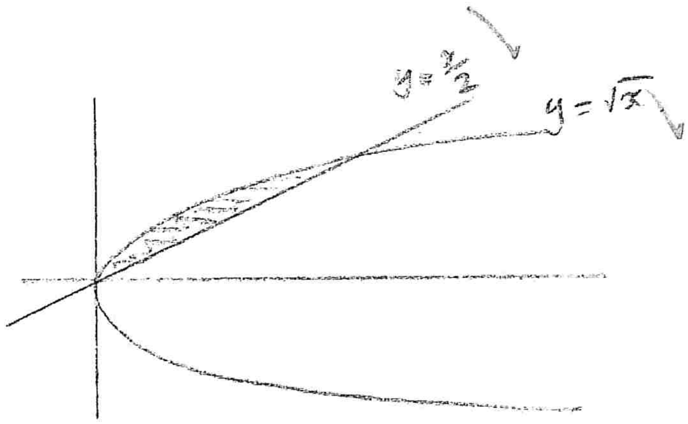
$$= -42\tfrac{2}{3}.$$

$$\int_1^5 x - 11 \, dx = \left[\frac{x^2}{2} - 11x \right]_1^5 = \left(\frac{25}{2} - 55 \right) - \left(\frac{1}{2} - 11 \right)$$

$$= -32$$

$\therefore$ shaded area $= 42\tfrac{2}{3} - 32 = 10\tfrac{2}{3}.$

d).



$\sqrt{x} = \frac{x}{2}$

$\therefore x = \frac{x^2}{4}$

$\therefore x^2 - 4x = 0$

$\therefore x(x-4) = 0$

$\therefore x = 0, 4.$

$$\int_0^4 x^{\frac{1}{2}} \, dx = \left[\frac{2x^{3/2}}{3} \right]_0^4 = \frac{16}{3}.$$

area of triangle $= \frac{1}{2} \times 4 \times 2 = 4$

$\therefore$ shaded area $= \frac{16}{3} - 4 = 1\frac{1}{3}.$

3. a) $y = 2 - x - x^2$ A is $(0, 2)$

$\therefore \frac{dy}{dx} = -1 - 2x$ $\therefore$ gradient at $A = -1$

$y - 2 = -x$

~~at A $2 - x - x^2 = -1 - 2x$~~

$\therefore$ $\underline{y = 2 - x}$

$\therefore$ ~~$x^2 + x - 2 = 2x + 1$~~

$\therefore$ ~~$x^2 - x - 3 = 0$~~

$\therefore$ ~~$(x+\quad)(x-\quad) = 0$~~

b) $x^2 + x - 2 = 0$ $\therefore (x+2)(x-1) = 0$ $\therefore x = -2, 1$

$\int_0^1 2 - x - x^2 \, dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left(2 - \frac{1}{2} - \frac{1}{3}\right) = 1\frac{1}{6}$

area of triangle $= \frac{1}{2} \times 2 \times 2 = 2$

$\therefore$ shaded area $= 2 - 1\frac{1}{6} = \frac{5}{6}$

4. ~~Section 2~~

a) VOLUME $= 2x^2 h = 4000$ $\therefore$ $h = \frac{2000}{x^2}$

b) area $= 2x^2 + 4xh + 2xh = 2x^2 + 6xh$

$\therefore A = 2x^2 + 6x \times \frac{2000}{x^2}$

$\therefore A = 2x^2 + \frac{12000}{x}$

c) $\frac{dA}{dx} = 4x - \frac{12000}{x^2}$

$\frac{dA}{dx} = 0$ for $4x - \frac{12000}{x^2} = 0$

$\therefore x^3 = 3000. \therefore \underline{\underline{x = 3000^{1/3} \simeq 14.4}}$

d) $A = 2 \times 3000^{2/3} + \frac{12000}{3000^{1/3}} = \underline{\underline{1248 \text{ cm}^2}}$

e) $\frac{d^2A}{dx^2} = 4 + \frac{24000}{x^3}$

$x = 3000^{1/3} \therefore \frac{d^2A}{dx^2} = 4 + \frac{24000}{3000} = 12 > 0$

$\underline{\underline{\therefore \text{a minimum}}}$

5. a) Surface area $= 2\pi rh + 2\pi r^2$ since closed at both ends

$= 30000$

$\therefore h = \frac{30000 - 2\pi r^2}{2\pi r}$

$V = \pi r^2 h = \pi r^2 \left(\frac{30{,}000 - 2\pi r^2}{2\pi r} \right)$

$= r(15000 - \pi r^2)$

$\therefore \underline{\underline{V = 15000r - \pi r^3}}$

b) $\frac{dV}{dr} = 15000 - 3\pi r^2$

at maximum $15000 - 3\pi r^2 = 0$

$\therefore r = \sqrt{\frac{5000}{\pi}}$

$\therefore V = 15000\sqrt{\frac{5000}{\pi}} - \pi \times \left(\frac{5000}{\pi}\right)^{3/2}$

$\therefore \underline{\underline{V = 398942 \text{ cm}^3}}$

$\frac{d^2V}{dr^2} = -6\pi r$

$r = \sqrt{\frac{5000}{\pi}} \therefore \frac{d^2V}{dr^2} < 0 \therefore \underline{\underline{\text{max.}}}$

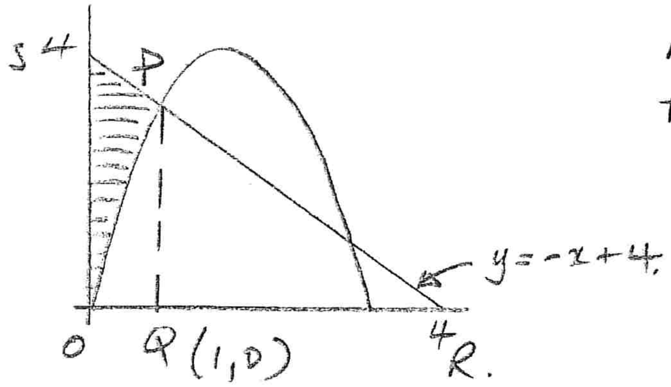
SECTION 3.

1. a) $y = 5x - 2x^2 \therefore \frac{dy}{dx} = 5 - 4x$

$\therefore$ at P $\frac{dy}{dx} = 5 - 4 \times 1 = 1$

$\therefore$ gradient of normal $= -1$

$y - 3 = -(x-1) \quad \therefore \underline{\underline{y = -x + 4}}$

b) 

required area = triangle OSR − triangle PQR − area under curve from O to Q

$$\text{area} = \tfrac{1}{2} \times 4 \times 4 - \tfrac{1}{2} \times 3 \times 3 - \int_0^1 5x - 2x^2 \, dx$$

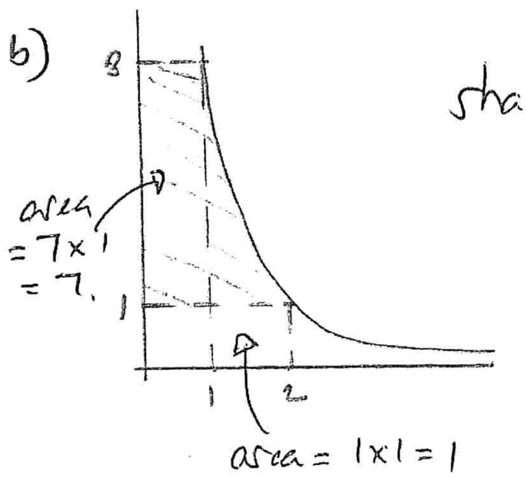
$$= 8 - \tfrac{9}{2} - \left[\frac{5x^2}{2} - \frac{2x^3}{3} \right]_0^1$$

$$= \tfrac{7}{2} - \left[\tfrac{5}{2} - \tfrac{2}{3} \right] = \underline{\underline{1\tfrac{2}{3} \text{ or } \tfrac{5}{3}}}$$

2. a) $$\int_1^2 \frac{8}{x^3} \, dx = \int_1^2 8x^{-3} \, dx = \left[-4x^{-2} \right]_1^2$$

$$= (-1) - (-4)$$

$$= \underline{\underline{3}}$$

b) 

$$\text{shaded area} = \int_1^2 \frac{8}{x^3} \, dx - 1 + 7$$

$$= \underline{\underline{9}}$$

3. a) area of cross-section of prism $= \frac{1}{2} \times x^2 \sin 60 = \frac{\sqrt{3}}{4} x^2$

$\therefore$ volume $= \frac{\sqrt{3}}{4} x^2 l = 250$

$$\therefore \quad l = \frac{1000}{\sqrt{3}\,.\,x^2}$$

b) $S.A. = 2\frac{\sqrt{3}}{4} x^2 + 3lx$

$$= \frac{\sqrt{3}}{2} x^2 + 3x \, . \, \frac{1000}{\sqrt{3}x^2} = \frac{\sqrt{3}}{2} x^2 + \frac{1000\sqrt{3}}{x}$$

$$A = \frac{\sqrt{3}}{2}\left(x^2 + \frac{2000}{x}\right)$$

c) $$\frac{dA}{dx} = \frac{\sqrt{3}}{2}\left(2x - \frac{2000}{x^2}\right)$$

$\frac{dA}{dx}$ is a minimum when $2x = \frac{2000}{x^2}$ $\quad \therefore \; x = 10$

d) $$A = \frac{\sqrt{3}}{2}\left(100 + \frac{2000}{10}\right) = 150\sqrt{3}$$

e) $$\frac{d^2A}{dx^2} = \frac{\sqrt{3}}{2}\left(2 + \frac{4000}{x^3}\right)$$

at $x = 10$ $\quad \frac{d^2A}{dx^2} = \frac{\sqrt{3}}{2}\left(2 + \frac{4000}{1000}\right) = 3\sqrt{3} > 0$

$\therefore$ a minimum