1.

Solutions to Pure 28 – Proof

Section 1

1 a) $a_1 = 2^1 + 1 = 3$ $\quad a_2 = 2^2 + 1 = 5$ $\quad a_4 = 2^4 + 1 = 17$

b) $a_3 = 2^3 + 1 = 9$ $\quad$ 9 is not prime $\therefore$ the sequence doesn't always produce prime numbers.

2 $a = 2$ $\quad d = 0.5$ $\quad u_n = 10 = 2 + (n-1) \times 0.5$

$\frac{8}{0.5} + 1 = n = 17$

$S_{17} = \frac{17}{2}(2 + 10) = 102$, $\quad 102 + (3 \times 10) = 132$ km.

3 a) $a = 3000$ $\quad r = 1.04$ $\quad n = 7$ $\quad u_7 = 3000 \times 1.04^6 = £3795.96$

b) $6000 = 3000 \times 1.04^n$

$2 = 1.04^n \Rightarrow \frac{\log 2}{\log 1.04} = n = 17.67\ldots$ so after 18 full years it will have doubled.

4. $S_\infty = \frac{a}{1-r} = 2a \Rightarrow \frac{1}{1-r} = 2 \Rightarrow 1 - r = \frac{1}{2} \Rightarrow r = \frac{1}{2}$

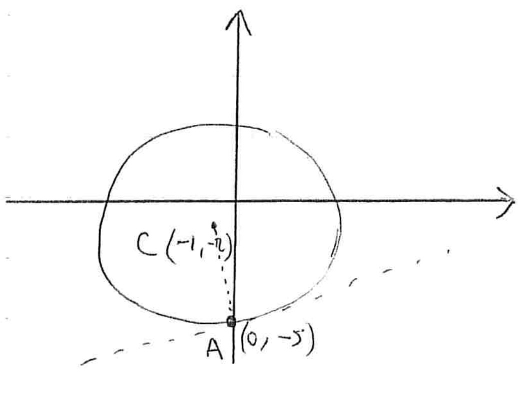
5. $f(3) = 0$ $\therefore x - 3$ is a factor of $f(x)$

$f(x) = (x-3)(ax^2 + bx + c)$ $\quad a = 2$ $\quad c = 2$, $b = 5$

$f(x) = (x-3)(2x^2 + 5x + 2) = (x-3)(2x+1)(x+2)$

6. $(x+1)^2 - 1 + (y+2)^2 - 4 = 5$

$(x+1)^2 + (y+2)^2 = 10$



Gradient of AC $= \frac{-5 - -2}{0 - -1} = \frac{-3}{1} = -3$

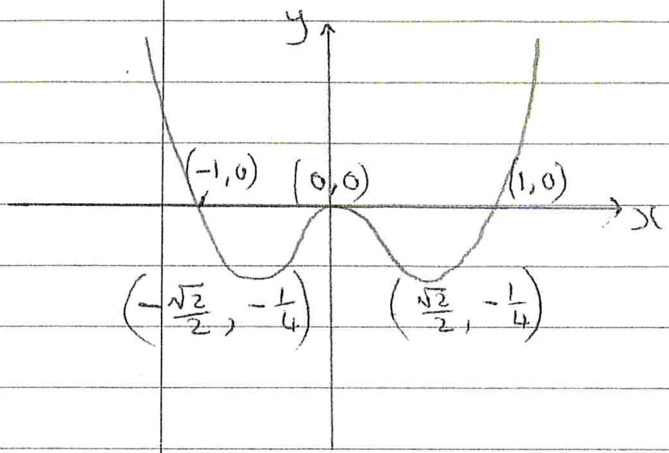
Gradient of tangent $= \frac{1}{3}$

$y - -5 = \frac{1}{3}(x - 0)$

$\frac{1}{3}x - y = 5$ $\quad$ or $\quad y - \frac{1}{3}x = -5$

or $\quad x - 3y = 15$

7) a) $y = x^4 - x^2 = x^2(x^2-1) = x^2(x+1)(x-1)$



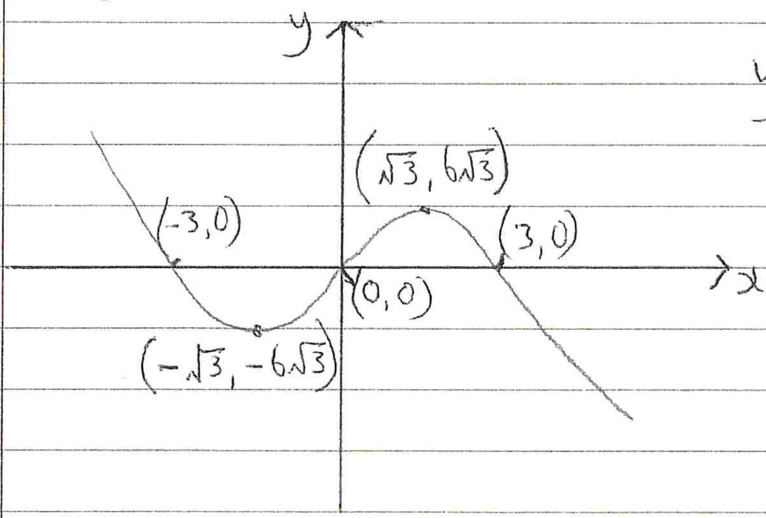
$y' = 4x^3 - 2x = 0$

$2x(2x^2-1) = 0$

$x = 0, \quad 2x^2 - 1 = 0 \quad x = \pm\frac{\sqrt{2}}{2}$

$y = -1/4.$

b) $y = -x^3 + 9x = x(-x^2+9) = x(3-x)(3+x)$



$y' = -3x^2 + 9 = 0$

$-3x^2 = -9$

$x^2 = 3$

$x = \pm\sqrt{3}$

$y = \pm 6\sqrt{3}$

Section 2:

1. Assume that M is the largest multiple of 3. M+3 is also a multiple of 3 and is larger than M. This contradicts the original assumption. ∴ there is no largest multiple of 3.

2. Assume there are two odd numbers $2n+1$ and $2m+1$ whose product is even. (n, m integer)
$(2n+1)(2m+1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$
This is an odd number which contradicts the assumption
$\therefore$ the product of two odd numbers is odd.

3. Let $n = 2k+1$, k integer.
$n^3+1 = (2k+1)^3+1 = 8k^3 + 12k^2 + 6k + 1 + 1 = 8k^3 + 12k^2 + 6k + 2$
$= 2(4k^3 + 6k^2 + 3k + 1)$ which must be even.
$\therefore$ if n is odd then n^3+1 will be even.

4. Assume that $\frac{a}{b}$ is the largest rational number where a and b are integer.
$\frac{a}{b} + 1 = \frac{a+b}{b}$ which is also rational and is larger than $\frac{a}{b}$

This contradicts the original assumption $\therefore$ there is no largest rational number.

5. Assume that there is some odd number n for which n^2 is even. $n = 2k+1 \Rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
n^2 is odd which contradicts the assumption.
$\therefore$ If n^2 is even then n must be even.

6. Assume that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{a}{b}$ (a, b integer)
Assume also that a and b have no common factors, otherwise $\frac{a}{b}$ could be simplified

$\sqrt{2} = \frac{a}{b} \Rightarrow b\sqrt{2} = a \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ is even
$\Rightarrow a$ is even. Let $a = 2n$ so $a^2 = 2b^2 \Rightarrow (2n)^2 = 2b^2$
$\Rightarrow 4n^2 = 2b^2 \Rightarrow 2n^2 = b^2 \Rightarrow b^2$ is even $\Rightarrow b$ is even
If a and b are both even, they have a common factor. (PTO)

This contradicts the original statement $\therefore \sqrt{2}$ is irrational.

7. Assume that $1+\sqrt{2}$ is rational so $1+\sqrt{2} = \frac{a}{b}$ for some integers a and b.

$$1+\sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2} = \frac{a}{b} - 1 \Rightarrow \sqrt{2} = \frac{a-b}{b} \Rightarrow \sqrt{2} \text{ rational}$$

$\sqrt{2}$ isn't rational so we have a contradiction.
$\therefore 1+\sqrt{2}$ is irrational.

8. $25a + 15b = 1 \Rightarrow 5(5a+3b) = 1 \Rightarrow 5a+3b = \frac{1}{5}$
This is not possible if a and b are integer
$\therefore$ no integers exist such that $25a + 15b = 1$

9. Assume that there is a finite number of primes say n, $P_1 = 2$ $P_2 = 3$ ..... P_{n-1} and P_n
Let $\mathbb{P} = P_1 P_2 P_3 \cdots P_{n-1} P_n$ (product of all primes)
$P+1$ will not be divisible by any number in $\mathbb{P}$ therefore $P+1$ is either prime or has prime factors not in $\mathbb{P}$, either of which is a contradiction. Therefore there are infinitely many primes.

10.
$$S_n = a + a+d + a+2d + \cdots + a+(n-2)d + a+(n-1)d$$
$$S_n = a+(n-1)d + a+(n-2)d + \cdots + a+d + a$$
$$2S_n = 2a+(n-1)d + 2a+(n-1)d + \cdots 2a+(n-1)d + 2a+(n-1)d$$
$$2S_n = n[2a+(n-1)d] \Rightarrow S_n = \frac{n}{2}[2a+(n-1)d]$$
as required.

11.
$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$
$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$
$$S_n - rS_n = a - ar^n$$
$$S_n(1-r) = a(1-r^n)$$
$$S_n = \frac{a(1-r^n)}{1-r}$$
as required.