Trig equations and proofs SOLUTIONS.

SECTION 1

1) $(2-3x)^5 = 2^5 + {}^5C_1(2)^4(-3x)^1 + {}^5C_2(2)^3(-3x)^2$

$= 32 - 240x + 720x^2 + \dots$

2)a) $(3+bx)^5 = 3^5 + {}^5C_1(3)^4(bx) + {}^5C_2(3)^3(bx)^2$

$= 243 + 405bx + 270b^2x^2 + \dots$

b) $2 \times 405b = 270b^2 \Rightarrow 270b^2 - 810b = 0$

$\Rightarrow b = 0$ or $\underline{\underline{b=3}}$

3a) $(1+\frac{x}{4})^8 = 1^8 + {}^8C_1(1)^7(\frac{x}{4}) + {}^8C_2(1)^6(\frac{x}{4})^2 + {}^8C_3(1)^5(\frac{x}{4})^3$

$= 1 + 2x + \frac{7}{4}x^2 + \frac{7}{8}x^3 + \dots$

b) $1 + \frac{x}{4} = 1.025 \Rightarrow \frac{x}{4} = 0.025 \Rightarrow x = 0.1$

$(1.025)^8 = 1 + 2(0.1) + \frac{7}{4}(0.1)^2 + \frac{7}{8}(0.1)^3 = 1.2184$

4a) $(2-9x)^4 = 2^4 + {}^4C_1(2)^3(-9x)^1 + {}^4C_2(2)^2(-9x)^2$

$= 16 - 288x + 1944x^2 + \dots$

b) $f(x) = (1+kx)(2-9x)^4 = (1+kx)(16 - 288x + 1944x^2 + \dots)$

$= A - 232x + Bx^2$

$A = 1 \times 16 = 16$

c) x terms: $1 \times -288 + 16k = -232 \Rightarrow 16k = 56 \Rightarrow k = \frac{7}{2}$

d) x^2 terms: $B = 1944 - 288k = 1944 - 1008 = 936$

Section 2

1a) $3\sec^2\theta = 4\tan^2\theta$

$\Rightarrow 3(1+\tan^2\theta) = 4\tan^2\theta$ ✓

$\Rightarrow \tan^2\theta = 3 \Rightarrow \tan\theta = \pm\sqrt{3}$ ✓ (4)

$\theta = \frac{\pi}{3}$ ✓, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ ✓

b) $\cot^2\theta - 3\mathrm{cosec}\,\theta + 3 = 0$

$\Rightarrow (\mathrm{cosec}^2\theta - 1) - 3\mathrm{cosec}\,\theta + 3 = 0$ ✓

$\Rightarrow (\mathrm{cosec}\,\theta - 1)(\mathrm{cosec}\,\theta - 2) = 0$ (4)

$\Rightarrow \mathrm{cosec}\,\theta = 1$ ✓ or $\mathrm{cosec}\,\theta = 2$

$\sin\theta = 1$ $\quad$ $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{2}$ ✓, $\frac{\pi}{6}$, $\frac{5\pi}{6}$ ✓

c) $1 + \tan^2\theta + 2\tan\theta = 0$ ✓ $\Rightarrow (\tan\theta + 1)^2 = 0$

$\tan\theta = -1$ ✓

$\theta = \cancel{-\frac{\pi}{4}}$, $\frac{3\pi}{4}$, $\frac{7\pi}{4}$ ✓ (3)

2a) $\sec^2 x - 1 - 2\sec x - 2 = 0 \Rightarrow \sec^2 x - 2\sec x - 3 = 0$

$(\sec x + 1)(\sec x - 3) = 0$ (5)

$\sec x = -1 \Rightarrow \cos x = -1 \quad x = 180°, -180°$

or $\sec x = 3 \Rightarrow \cos x = 1/3 \quad x = 70.5°, -70.5°$

b) $\operatorname{cosec}^2 x + 5\operatorname{cosec} x + 2(\operatorname{cosec}^2 x - 1) = 0$

$\Rightarrow 3\operatorname{cosec}^2 x + 5\operatorname{cosec} x - 2 = 0$

$(3\operatorname{cosec} x - 1)(\operatorname{cosec} x + 2) = 0$ (5)

$\operatorname{cosec} x = 1/3 \rightarrow$ no solutions

$\operatorname{cosec} x = -2 \Rightarrow \sin x = -1/2 \quad x = -150°, -30°$

c) $\sec^2 x - 1 + 4\sec x - 2 = 0 \Rightarrow \sec^2 x + 4\sec x - 3 = 0$

$\Rightarrow \sec x = -2 \pm \sqrt{7}$ (4)

$\Rightarrow \cos x = \frac{1}{-2 \pm \sqrt{7}} = -0.2153$ or $1.5486 \rightarrow$ no solution

$x = -102.4°, 102.4°$

3a) $\cot^2 2x + \operatorname{cosec} 2x - 1 = 0 \qquad 0 \le 2x \le 720$

$(\operatorname{cosec}^2 2x - 1) + \operatorname{cosec} 2x - 1 = 0$

$\operatorname{cosec}^2 2x + \operatorname{cosec} 2x - 2 = 0$

$(\operatorname{cosec} 2x + 2)(\operatorname{cosec} 2x - 1) = 0$

$\operatorname{cosec} 2x = -2 \qquad \operatorname{cosec} 2x = 1$ (6)

$\sin 2x = -1/2 \qquad \sin 2x = 1$

$2x = 210, 330, 570, 690, 90, 450$

$x = 45, 105, 165, 225, 285, 345$

b) $3\operatorname{cosec}^2 x - 4\sin^2 x = 1$

$\frac{3}{\sin^2 x} - 4\sin^2 x = 1$

$3 - 4\sin^4 x = \sin^2 x$

$4\sin^4 x + \sin^2 x - 3 = 0$

$(4\sin^2 x - 3)(\sin^2 x + 1) = 0$

$\sin^2 x = 3/4 \qquad \sin^2 x = -1 \rightarrow$ no solution

$\sin x = \pm\sqrt{3}/2$

$x = 60, 120, 240, 300$ (5)

4)a) $LHS = (1 + \cot^2 x) - (1 + \tan^2 x)$ ✓

$\equiv \cot^2 x - \tan^2 x = RHS$ ✓ (2)

b) $LHS = \cos^2 x - 4 + 4\sec^2 x$ ✓

$\equiv \cos^2 x - 4 + 4(1 + \tan^2 x)$ ✓

$\equiv \cos^2 x + 4\tan^2 x = RHS$ ✓ (3)

c) $LHS = (\tan x + \cot x)(\tan x + \cot x)$ ✓

$\equiv \tan^2 x + 1 + 1 + \cot^2 x$ ✓

$\equiv \sec^2 x + \operatorname{cosec}^2 x = RHS$ ✓ (3)

d) $LHS = \dfrac{1}{\cos^2 x} + \dfrac{1}{\sin^2 x}$ ✓

$\equiv \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x \times \sin^2 x}$ ✓

$\equiv \dfrac{1}{\cos^2 x \sin^2 x}$ ✓ (4)

$\equiv \sec^2 x \operatorname{cosec}^2 x = RHS$ ✓

5)a) $1 + \tan^2 x \equiv \sec^2 x \Rightarrow \sec^2 x - \tan^2 x \equiv 1$ ✓

$\Rightarrow (\sec x + \tan x)(\sec x - \tan x) \equiv 1$ ✓

$\Rightarrow (-3)(\sec x - \tan x) = 1$

$\Rightarrow \sec x - \tan x = -1/3$ ✓ (3)

b) $\sec x + \tan x = -3$
$\sec x - \tan x = -1/3$ } solving simultaneously ✓

$\sec x = -5/3$ ✓ $\tan x = -4/3$ ✓ (3)

c) If $\sec x = -5/3$ $\cos x = -3/5$ ✓ and $\tan x = -4/3$

$\therefore x$ is obtuse ✓

$\cos^{-1}(-3/5) = \underline{126.86}$ (or -126.86)

$\tan^{-1}(-4/3) = -53.1$ or $\underline{126.86}$

$x = 126.86$ ✓ (3)