PURE 14 SOLUTIONS

SECTION 1

1. $\sqrt{2^2+1^2+3^2} = \sqrt{14}$ $\qquad \underline{\underline{\frac{1}{\sqrt{14}}(2\underline{i}+\underline{j}-3\underline{k})}}$

2 a) $\vec{OA}+\vec{AB} = \vec{OB} = (2\underline{i}+5\underline{j}-4\underline{k})+(3\underline{i}-5\underline{j}-\underline{k})$

$\therefore \underline{\underline{\vec{OB} = 5\underline{i}-5\underline{k}}}$

$\underline{\underline{\vec{OC} = \underline{i}-3\underline{j}-2\underline{k}}}$

b) $\vec{OA}+\vec{AC} = \vec{OC}$

$\therefore \vec{AC} = \vec{OC}-\vec{OA} = (\underline{i}-3\underline{j}-2\underline{k})-(2\underline{i}+5\underline{j}-4\underline{k})$

$\therefore \underline{\underline{\vec{AC} = -\underline{i}-8\underline{j}+2\underline{k}}}$

c) $|\vec{AC}| = \sqrt{1^2+8^2+2^2} = \underline{\underline{\sqrt{69}}}$

d) $|\vec{OC}| = \sqrt{1^2+3^2+2^2} = \underline{\underline{\sqrt{14}}}$

3. $|\vec{AB}| = \sqrt{2^2+3^2+5^2} = \sqrt{38}$

$\theta_x = \cos^{-1}\frac{2}{\sqrt{38}} = \underline{\underline{71.1^\circ}}$

$\theta_y = \cos^{-1}\frac{3}{\sqrt{38}} = \underline{\underline{60.9^\circ}}$

$\theta_z = \cos^{-1}\frac{-5}{\sqrt{38}} = \underline{\underline{144.2^\circ}}$

4. $f'(x) = -6x^2 - 3$ so since $x^2 > 0$ for all x
$f'(x) < 0$ for all x
$\therefore$ $f(x)$ is decreasing for all x.

---

5. $f'(x) = 3px^2 - 6px + 2x$

$f''(x) = 6px - 6p + 2$

$f''(2) = 12p - 6p + 2 = -1$

$\therefore \quad p = -\frac{1}{2}$

---

6. $fg(x) = (2x+5)^2 = 4x^2 + 20x + 25 = 9$

$\therefore \quad 4x^2 + 20x + 16 = 0$

$\therefore \quad x^2 + 5x + 4 = 0$

$\therefore \quad (x+1)(x+4) = 0$

$\therefore \quad x = -1, -4$

---

7. let $y = \frac{1}{x} - 3 \quad \therefore \quad \frac{1}{x} = y + 3 \quad \therefore \quad x = \frac{1}{y+3}$

$\therefore \quad f^{-1}(x) = \frac{1}{x+3}$

$f(2) = -\frac{5}{2} \qquad f(5) = -\frac{14}{5}$

domain of inverse = range of function

$\therefore \quad f^{-1}(x) = \frac{1}{x+3}, \quad x \in \mathbb{R}, \quad -\frac{14}{5} < x < -\frac{5}{2}$

---

8 $(x-5)^2 = x^2 - 10x + 25$

$(y-4)^2 = y^2 - 8y + 16$

$\therefore (x-5)^2 + (y-4)^2 = x^2 + y^2 - 10x - 8y + 41$

$\therefore (x-5)^2 + (y-4)^2 - 20 = x^2 + y^2 - 10x - 8y + 21 = 0$

$\therefore (x-5)^2 + (y-4)^2 = 20$

$\therefore$ centre is $(5, 4)$

circle cuts x-axis at $y = 0$ $\therefore x^2 - 10x + 21 = 0$

$\therefore (x-3)(x-7) = 0$

$\therefore x = 3, 7$.

gradient of radius at $x = 3$ is $\frac{4-0}{5-3} = 2$

$\therefore$ gradient of tangent at $x = 3$ is $-\frac{1}{2}$.

$\therefore$ equation of tangent at $x = 3$ is $\underline{\underline{y = -\frac{1}{2}(x-3)}}$

gradient of radius at $x = 7$ is $\frac{4-0}{5-7} = -2$

$\therefore$ gradient of tangent at $x = 7$ is $\frac{1}{2}$.

$\therefore$ equation of tangent at $x = 7$ is $\underline{\underline{y = \frac{1}{2}(x-7)}}$

9. $\log(y-x) = 0 \quad \therefore \; y - x = 1$ (the base doesn't matter)

$\therefore \; y = x + 1$

$\therefore \; 2\log(x+1) = \log(21+x)$

$\therefore \; (x+1)^2 = x + 21$

$\therefore \; x^2 + x - 20 = 0$

$\therefore \; (x+5)(x-4) = 0$

$\therefore \; x = -5, 4$

but cannot have $x = -5$ since have $\log(x+1) \; \therefore \; x + 1 > 0$

$\therefore \; \underline{\underline{x = 4 \quad y = 5}}$

---

10. let $y = 2^x \quad \therefore \; y^2 = 2^{2x}$

$\therefore \; y^2 - y - 6 = 0$

$\therefore \; (y-3)(y+2) = 0$

$\therefore \; y = 3, -2$

$\therefore \; 2^x = 3, -2$ but -2 cannot be a solution

$\therefore \; x \ln 2 = \ln 3$

$\therefore \; \underline{\underline{x = \frac{\ln 3}{\ln 2}}} \; (\simeq 1.58)$ (or $x = \log_2 3$)

SECTION 2.

1 a) $$\frac{5\left(1-\frac{1}{2}(2\theta)^2\right) - 3\theta - 4}{1-5\theta}$$

$$= \frac{5-10\theta^2-3\theta-4}{1-5\theta}$$

$$= \frac{(1-5\theta)(2\theta+1)}{1-5\theta} = \underline{\underline{2\theta+1}}$$

b) for small $\theta \Rightarrow \underline{\underline{1}}$

---

2. $$\frac{(3\theta)^2}{1-\left(1-\frac{1}{2}(2\theta)^2\right)} = \frac{9\theta^2}{2\theta^2} = \underline{\underline{4.5}}$$

---

3. $$\frac{(5\theta)^2+2\theta}{\theta} \simeq 3$$

$$\therefore 25\theta + 2 \simeq 3 \quad \therefore \underline{\underline{\theta \simeq \tfrac{1}{25}}}$$

---

4 a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

4a) (cont)

$\sin\theta \approx \theta$ for small θ $\therefore$ $\dfrac{\sin\theta}{\theta} \approx 1$

$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for small θ $\therefore$ $\dfrac{\cos\theta - 1}{\theta} \approx -\frac{1}{2}\theta$

$\therefore$ $f'(x) = \lim\limits_{h \to 0} \sin x\left(-\frac{1}{2}h\right) + \cos x\,(1)$

$\therefore$ $\underline{\underline{f'(x) = \cos x}}$

4b) $f(x) = \cos 3x$ $\therefore$ $f(x+h) = \cos 3(x+h) = \cos(3x+3h)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(3x+3h) - \cos 3x}{h}$$

$$= \lim_{h \to 0} \frac{\cos 3x \cos 3h - \sin 3x \sin 3h - \cos 3x}{h}$$

$$= \lim_{h \to 0} \cos 3x\left(\frac{\cos 3h - 1}{h}\right) - \sin 3x\left(\frac{\sin 3h}{h}\right)$$

for small θ $\sin\theta \approx \theta$ $\therefore$ $\sin k\theta \approx k\theta$ $\therefore$ $\dfrac{\sin k\theta}{\theta} \approx k$

$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ $\therefore$ $\cos k\theta \approx 1 - \frac{1}{2}(k\theta)^2$

4b) (cont) $\therefore \dfrac{\cos k\theta - 1}{\theta} \approx -\frac{1}{2}k^2\theta$

$$\therefore f'(x) = \lim_{h \to 0} \cos 3x\left(-\tfrac{1}{2}\times 3^2 h\right) - \sin 3x\,(3)$$

$$\therefore \underline{\underline{f'(x) = -3\sin 3x}}$$

4c) $f(x) = 4\cos x + 3x^2 \quad \therefore f(x+h) = 4\cos(x+h) + 3(x+h)^2$

$$\therefore f'(x) = \lim_{h \to 0} \frac{4\cos(x+h) + 3(x+h)^2 - 4\cos x - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{4\cos x\cos h - 4\sin x\sin h + 3x^2 + 3h^2 + 6xh - 4\cos x - 3x^2}{h}$$

$$= \lim_{h \to 0} 4\cos x\left(\frac{\cos h - 1}{h}\right) - 4\sin x\left(\frac{\sin h}{h}\right) + \frac{3h^2 + 6xh}{h}$$

for small θ $\sin\theta \approx \theta \therefore \dfrac{\sin\theta}{\theta} \approx 1$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2 \therefore \dfrac{\cos\theta - 1}{\theta} \approx -\frac{1}{2}\theta$

$$\therefore f'(x) = \lim_{h \to 0} 4\cos x\left(-\tfrac{1}{2}h\right) - 4\sin x\,(1) + 3h + 6x$$

$$\therefore \underline{\underline{f'(x) = -4\sin x + 6x}}$$

(27 MARKS)