SECTION 1

1. $f(x) = g(x) \quad \therefore \quad 2x - 10 = x^2 - 9$

$$\therefore \quad x^2 - 2x + 1 = 0$$

$$\therefore \quad (x-1)(x-1) = 0$$

$$\therefore \quad \underline{\underline{x = 1}}$$

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2. $y = ax^2 + bx + c \quad \therefore \frac{dy}{dx} = 2ax + b$

at the minimum $\frac{dy}{dx} = 0 \quad \therefore \quad x = \frac{-b}{2a} \quad \therefore \frac{-b}{2a} = 5$

$$\therefore \quad b = -10a.$$

at the minimum $y = 25a + 5b + c = -3$

$$\therefore \quad 25a - 50a + c = -3$$

$$\therefore \quad c = 25a - 3.$$

graph passes through $(4, 0)$

$$\therefore \quad 16a + 4b + c = 0$$

$$\therefore \quad 16a - 40a + 25a - 3 = 0$$

$$\therefore \quad a = 3$$

$b = -10a = -30 \qquad c = 25a - 3 = 72$

$$\underline{\underline{a = 3, \quad b = -30, \quad c = 72}}$$

3 $\quad x + y = 3 \qquad \therefore \; y = 3 - x$

$\quad\; x^2 - 3y = 1$

$\therefore \; x^2 - 3(3 - x) = 1$

$\therefore \; x^2 + 3x - 10 = 0$

$\therefore \; (x + 5)(x - 2) = 0$

$\therefore \quad x = -5, \, 2$

$x = -5 \qquad \therefore \; y = 3 - -5 = 8$

$x = 2 \qquad \therefore \; y = 3 - 2 = 1$

$\underline{\text{Hence } (-5, 8) \text{ and } (2, 1)}$

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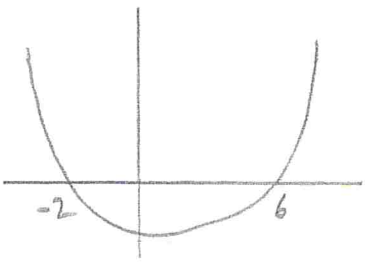
4. $\quad 12 + 4x > x^2$

$\therefore \; x^2 - 4x - 12 < 0$

$\therefore \; (x - 6)(x + 2) < 0$

$\therefore$ critical values are $x = 6, -2$



$\therefore \; \underline{-2 < x < 6}$

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5. $\dfrac{5}{x-3} < 2$

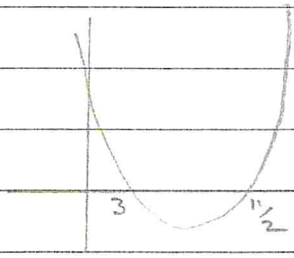
$\therefore\ 5(x-3) < 2(x-3)^2$

$\therefore\ 2(x-3)^2 - 5(x-3) > 0$

$\therefore\ 2x^2 - 17x + 33 > 0$

$\therefore\ (2x-11)(x-3) > 0$

$\therefore$ critical values are $x = 11/2,\ 3.$



$x < 3$ or $x > 11/2$

6. $2kx^2 + 5kx + 5k - 3 = 0$

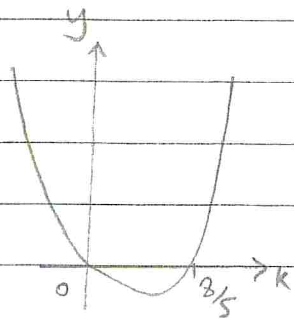
real roots $\Rightarrow$ discriminant ≥ 0

$\therefore\ (5k)^2 - 4(2k)(5k-3) \geq 0$

$\therefore\ 15k^2 - 24k \leq 0$

$\therefore\ 3k(5k-8) \leq 0$

$\therefore$ critical values are $k = 0,\ 8/5.$



$\therefore\ 0 < k \leq 8/5$ (since $k \neq 0$)

7. a) $(6, 1)$ b) $(2, 3)$ c) $(2, -2)$

8. $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

$\therefore (3+bx)^5 = \sum_{r=0}^{5} \binom{5}{r} 3^{5-r} (bx)^r$

for coefficient of x^3, $r = 3$

$\therefore \binom{5}{3} 3^2 b^3 = -720$

$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \times 4 \times 3!}{3! \times 2} = 10$

$\therefore 10 \times 3^2 b^3 = -720$

$\therefore \underline{\underline{b = -2}}$

9. $(2k+x)^n = (2k)^n + \binom{n}{1}(2k)^{n-1}x + \binom{n}{2}(2k)^{n-2}x^2$
$+ \binom{n}{3}(2k)^{n-3}x^3 + \ldots + x^n$

$\binom{n}{2}(2k)^{n-2} = \binom{n}{3}(2k)^{n-3}$

$\therefore \frac{n!}{(n-2)!\,2!} = \frac{n!}{(n-3)!\,3!} \times \frac{1}{2k}$

$\therefore\ 2k\,(n-3)!\,3! = (n-2)!\,2!$

$(n-3)! = (n-3)(n-4)(n-5)\ldots$

$(n-2)! = (n-2)(n-3)(n-4)\ldots$

$\therefore\ (n-2)! = (n-2)(n-3)!$

$\therefore\ 6k\,(n-3)! = (n-2)(n-3)!$

$\therefore\ 6k = n-2$

$\therefore\ \underline{\underline{n = 6k+2}}$

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10. $\left(2+\frac{x}{5}\right)^{10} = 2^{10} + \binom{10}{1}2^{9}\left(\frac{x}{5}\right) + \binom{10}{2}2^{8}\left(\frac{x}{5}\right)^2 + \binom{10}{3}2^{7}\left(\frac{x}{5}\right)^3 + \ldots$

$= 1024 + 1024x + \frac{2304}{5}x^2 + \frac{3072}{25}x^3 + \ldots$

$\left(2+\frac{x}{5}\right)^{10} = 2.1^{10}$ for $x = 0.5$

$\therefore\ 2.1^{10} \simeq 1024 + 1024\times 0.5 + \frac{2304}{5}\times 0.5^2 + \frac{3072}{25}\times 0.5^3$

$= \underline{\underline{1666.56}}$

$\%\ \text{error} = \frac{2.1^{10} - 1666.56}{2.1^{10}} \times 100 = \frac{1667.99 - 1666.56}{1667.99} \times 100$

$= \underline{\underline{0.09\% < 0.1\%}}$

SECTION 2

1. a) $0^3 - 9(0) + 1 = 1$ sign change $\therefore$ root
$1^3 - 9(1) + 1 = -7$ between $x = 0, 1$.

b) $e^{-10} - 9\cos 40 = 6.002$ sign change $\therefore$ root
$e^{-11} - 9\cos 44 = -8.999$ between $x = 10, 11$.

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2. $x_{n+1} = \frac{1}{3}(x_n^3 + 1) \Rightarrow x = \frac{1}{3}(x^3 + 1)$

$\therefore 3x = x^3 + 1$

$\therefore \underline{x^3 - 3x + 1 = 0}$

$x_{n+1} = \dfrac{3x_n - 1}{x_n^2} \Rightarrow x = \dfrac{3x - 1}{x^2}$

$\therefore x^3 = 3x - 1$

$\therefore \underline{x^3 - 3x + 1 = 0}$

using $x_{n+1} = \frac{1}{3}(x_n^3 + 1)$:

$x_0 = 0.2 \therefore x_1 = \frac{1}{3}(0.2^3 + 1) = 0.336$
$x_2 = \frac{1}{3}(0.336^3 + 1) = 0.3459776853$
$x_3 = 0.3471379074$
$x_4 = 0.3472772529$
$x_5 = 0.3472940514$
$x_6 = 0.3472960775$
$x_7 = 0.3472963218$
$x_8 = 0.3472963513$
$x_9 = 0.3472963548$

Hence, Solution converges to $\underline{\underline{x = 0.3472964}}$ to 7 decimal places.

using $x_{n+1} = \dfrac{3x_n - 1}{x_n^2}$

$x_0 = 0.2 \therefore x_1 = \dfrac{3 \times 0.2 - 1}{0.2^2} = -10$

$x_2 = \dfrac{3 \times -10 - 1}{(-10)^2} = -0.31$

$x_3 = -20.08324662$

$x_4 = -0.1518575559$

Clearly, this solution is $\underline{\text{not}}$ converging.

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3. a) $g(x) = e^{x-1} + x - 6$

$g(x) = 0 \therefore e^{x-1} + x - 6 = 0$

$\therefore e^{x-1} = 6 - x$

$\therefore \ln e^{x-1} = \ln(6-x)$

$\therefore x - 1 = \ln(6-x)$

$\therefore \underline{\underline{x = \ln(6-x) + 1}}$

(and $x < 6$ to make $\ln(6-x)$ real)

3b) $x_1 = \ln(6 - x_0) + 1 = \ln 4 + 1$

$\therefore x_1 = 2.386294361$ $\qquad \underline{\underline{x_1 = 2.3863}}$

$x_2 = \ln(6 - 2.386294361) + 1$

$\therefore x_2 = 2.284733739$ $\qquad \underline{\underline{x_2 = 2.2847}}$

$x_3 = \ln(6 - 2.284733739) + 1$

$\therefore x_3 = 2.312450348$ $\qquad \underline{\underline{x_3 = 2.3125}}$

c) $g(2.3075) = e^{1.3075} + 2.3075 - 6 = 0.004$

$g(2.3065) = e^{1.3065} + 2.3065 - 6 = -0.0003$

Sign change $\therefore$ there is a root between $x = 2.3065$ and $x = 2.3075$ hence $\alpha = 2.307$ to 3 d.p.

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4. a) $f(x) = 2\sin x^2 + x - 2 \qquad 0 \leq x < 2\pi$

$2\sin 0.75^2 + 0.75 - 2 = -0.1834$

$2\sin 0.85^2 + 0.85 - 2 = 0.1725$

Sign change $\therefore$ root between $x = 0.75, 0.85$.

b) $2\sin x^2 + x - 2 = 0 \quad \therefore \sin x^2 = 1 - 0.5x$

$\therefore x^2 = \sin^{-1}(1 - 0.5x) \quad \therefore \underline{\underline{x = \left(\sin^{-1}(1-0.5x)\right)^{\frac{1}{2}}}}$

4c) $x_1 = \left(\sin^{-1}(1 - 0.5 \times 0.8)\right)^{\frac{1}{2}}$

$\therefore x_1 = 0.8021852085$ $\qquad x_1 = 0.80219$

$x_2 = 0.8013339203$ $\qquad x_2 = 0.80133$

$x_3 = 0.8016655593$ $\qquad x_3 = 0.80167$

d) $f(0.801575) = 2\sin 0.801575^2 + 0.801575 - 2$
$= 0.00000862$

$f(0.801565) = 2\sin 0.801565^2 + 0.801565 - 2$
$= -0.000027$

Sign change $\therefore \alpha = 0.80157$ is a root to 5 decimal places.

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5. a) $f(x) = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 5$

$f'(x) = x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

b) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = 5 - \dfrac{(2(5)^{\frac{1}{2}} + 5^{-\frac{1}{2}} - 5)}{5^{-\frac{1}{2}} - \frac{1}{2}(5)^{-\frac{3}{2}}}$

$\therefore x_1 = 5.200377653$ $\therefore$ to 2 s.f. $\alpha = 5.2$

5c) $f'(0.5) = 0$ $\therefore$ tangent line will not intersect the x-axis (or will get division by zero)

$$\frac{f(x)}{f'(x)} = \frac{2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 5}{x^{-\frac{1}{2}} - \frac{1}{2}x^{-3/2}} = \frac{2x^2 + x - 5x^{3/2}}{x - \frac{1}{2}}$$

$$\therefore \frac{f(0)}{f'(0)} = 0$$

$\therefore$ $x_{n+1} = x_n$ $\therefore$ will not iterate to a solution.

d) $x_0 = 0.1$ $\quad x_1 = 0.1 - \dfrac{2(0.1)^{\frac{1}{2}} + (0.1)^{-\frac{1}{2}} - 5}{(0.1)^{-\frac{1}{2}} - \frac{1}{2}(0.1)^{-3/2}}$

$= 0.00471529\ldots$

$x_2 = 0.0110567\ldots$

$x_3 = 0.0222811\ldots$

$x_4 = 0.0361902\ldots$

$x_5 = 0.0456467\ldots$

$x_6 = 0.0479613\ldots$

$x_7 = 0.0480588\ldots$

Converges to 2 s.f.

$x_7 = 0.048$ to 2 s f $\therefore$ $x_6 = x_7$ to 2 s f

$\therefore$ $\underline{\beta = 0.048}$

6. $f(x) = x^2 - 612 \quad \therefore \quad f'(x) = 2x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 25 - \frac{25^2 - 612}{2 \times 25} = \underline{\underline{24.74}}$$

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7 a) $f(x) = x^3 - 612 \quad \therefore \quad f'(x) = 3x^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

choose $x_0 = 8 \quad \therefore \quad x_1 = 8 - \frac{512 - 612}{3 \times 64} = \underline{\underline{8.52}}$

(choosing $x_0 = 9$ also leads to this result).

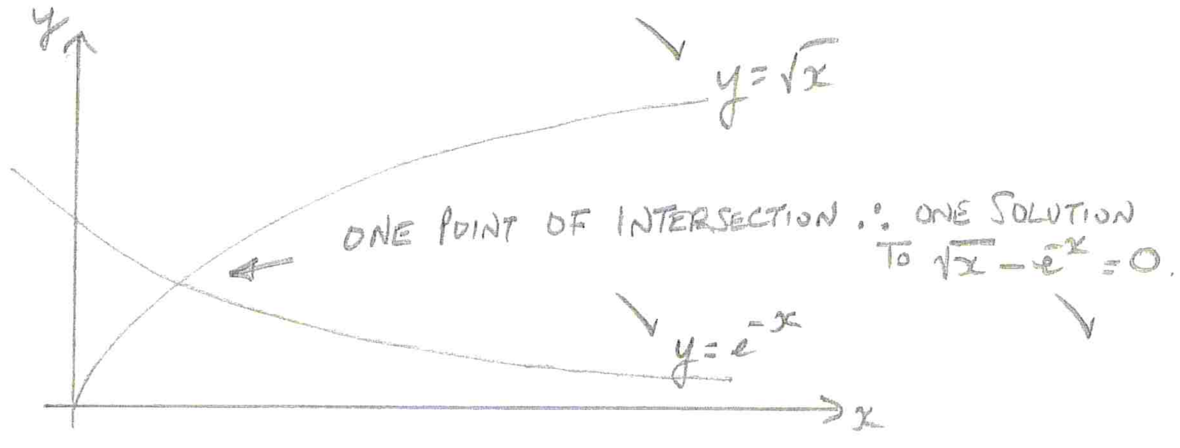
b) $f(x) = x^{10} - 612 \quad \therefore \quad f'(x) = 10x^9$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

choose $x_0 = 2 \quad \therefore \quad x_1 = 2 - \frac{2^{10} - 612}{10 \times 2^9} = \underline{\underline{1.92}}$

(NB. choosing $x_0 = 1$ leads to wild fluctuations)

8.



$$f(x) = x^{1/2} - e^{-x} \quad \therefore \; f'(x) = \tfrac{1}{2}x^{-1/2} + e^{-x}$$

$$x_1 = x_0 - \frac{x^{1/2} - e^{-x}}{\tfrac{1}{2}x^{-1/2} + e^{-x}}$$

$$x_0 = 1 \quad \therefore \; x_1 = 1 - \frac{1 - e^{-1}}{\tfrac{1}{2} + e^{-1}} = 0.271649\ldots$$

$$\underline{\underline{x_1 = 0.272}}$$

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9. $$f(t) = \ln(t+2) + \sin 0.5t + t^{1/2}$$

$$\therefore \; f'(t) = \frac{1}{t+2} + 0.5\cos 0.5t + \tfrac{1}{2}t^{-1/2}$$

$$t_0 = 12 \qquad f(t) = 7$$

$$t_1 = 12 - \frac{\ln 14 + \sin 6 + \sqrt{12} - 7}{\tfrac{1}{14} + 0.5\cos 6 + \tfrac{1}{2} \times 12^{-1/2}}$$

$$= 13.69038\ldots$$

$\Rightarrow$ <u>13 years 9 months.</u>

TOTAL 67