Pure 1 solutions - section 1

1 a) i) $3x - 5 < 11 - x$

$4x < 16$

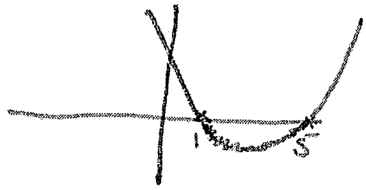
$x < 4$

ii) $x^2 - 6x + 5 \leq 0$

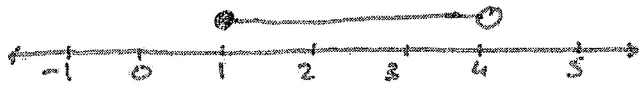
$(x-5)(x-1) \leq 0$

critical values: 5, 1

$1 \leq x \leq 5$



b)



2) a) $y = ab^x$

$(0, 5)$: $5 = ab^0 \Rightarrow 5 = a$

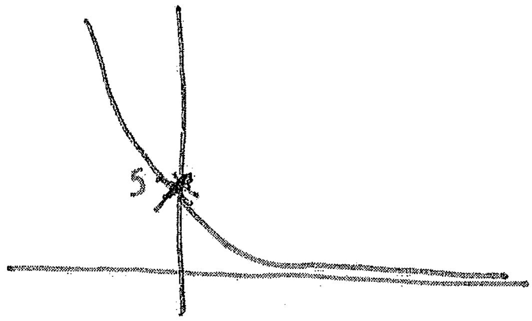
$(2, 1.25)$: $1.25 = 5b^2$

$0.25 = b^2$

$0.5 = b$

$y = 5(0.5)^x$

b)



3 Prove: if n is an integer, and n^n is odd, then n is odd

Assume: if n is an integer and n^n is odd, then n is even

if n is even it can be written as $2p$

$\therefore n^n = (2p)^{2p} = (4p^2)^p = 4(4^{p-1}p^{2p})$ = even no.

Contradiction: since n is even n^n is also even, this contradicts the assumption that n^n is odd and proves the original statement

4 a) $u_2 = 120, \ u_5 = 15$

$u_n = ar^{n-1}$

$\therefore \ 120 = ar$ and $15 = ar^4$

$15 = 120r^3$

$\frac{1}{8} = r^3$

$\frac{1}{2} = r$

b) $120 = a \times \frac{1}{2}$

$240 = a$

c) $S_\infty = \frac{a}{1-r}$

$= \frac{240}{1-\frac{1}{2}}$

$= 480$

5 $x = 5\sec\theta$ $\qquad y = 3\tan\theta$

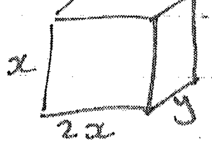
$\sec\theta = \frac{x}{5}$ $\qquad \tan\theta = \frac{y}{3}$

$1 + \tan^2\theta \equiv \sec^2\theta$

$1 + \left(\frac{y}{3}\right)^2 = \left(\frac{x}{5}\right)^2$

$\therefore \ \frac{x^2}{25} - \frac{y^2}{9} = 1$

6) a) $V = 2x^2y = 192 \quad \therefore y = \frac{96}{x^2}$



~~SA~~ $\cancel{S} = (2x^2 + xy + 2xy) \times 2$

$\cancel{S} = 4x^2 + 6xy$

$S = 4x^2 + 6x \times \frac{96}{x^2}$

$S = 4x^2 + \frac{576}{x}$

b) $\frac{dS}{dx} = 8x - 576x^{-2} = 0$

$8x = \frac{576}{x^2}$

$x^3 = 72$

$x = \sqrt[3]{72} = 4.16$ (3 s f)

$\therefore$ Min $S = 4(x)^2 + \frac{576}{x} = 208$ (3s.f.)

c) $\frac{d^2S}{dx^2} = 8 + 1152x^{-3}$

$\frac{d^2S}{dx^2} = 8 + 1152(x)^{-3}$

$= 24$

second derivative is positive, therefore point is a minimum

Pure 19 section 2

1

a $u = 2x - 1 \;\therefore\; x = \frac{1}{2}(u+1),\; \frac{du}{dx} = 2$

$$\int x(2x-1)^4\,dx = \int \tfrac{1}{2}(u+1)u^4 \times \tfrac{1}{2}\,du$$
$$= \tfrac{1}{4}\int (u^5 + u^4)\,du \checkmark$$
$$= \tfrac{1}{4}(\tfrac{1}{6}u^6 + \tfrac{1}{5}u^5) + c$$
$$= \tfrac{1}{4}[\tfrac{1}{6}(2x-1)^6 + \tfrac{1}{5}(2x-1)^5] + c \checkmark$$
$$= \tfrac{1}{120}(2x-1)^5[5(2x-1)+6] + c$$
$$= \tfrac{1}{120}(10x+1)(2x-1)^5 + c \checkmark$$

b $u = 1 - x \therefore x = 1 - u, \frac{dx}{du} = -1$

$$\int x\sqrt{1-x}\,dx = \int (1-u)\times\sqrt{u}\times -1\,du$$
$$= -\int u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$$
$$= -\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + c \checkmark$$
$$= -\left(\frac{2}{3}(1-x)^{\frac{3}{2}} - \frac{2}{5}(1-x)^{\frac{5}{2}}\right) + c \checkmark$$
$$= -\frac{2}{15}(1-x)^{\frac{3}{2}}(5 - 3(1-x)) + c$$
$$= -\frac{2}{15}(2+3x)(1-x)^{\frac{3}{2}} + c \checkmark$$

c $x = \sin u \;\therefore\; \frac{dx}{du} = \cos u$

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \int \frac{1}{\cos^3 u}\times\cos u\,du \checkmark$$
$$= \int \sec^2 u\,du$$
$$= \tan u + c \checkmark$$
$$= \frac{\sin u}{\cos u} + c$$
$$= \frac{x}{\sqrt{1-x^2}} + c \checkmark$$

d $x = u^2 \;\therefore\; \frac{dx}{du} = 2u$

$$\int \frac{1}{\sqrt{x}-1}\,dx = \int \frac{1}{u-1}\times 2u\,du \checkmark$$
$$= \int \frac{2(u-1)+2}{u-1}\,du$$
$$= \int \left(2 + \frac{2}{u-1}\right)du \checkmark$$
$$= 2u + 2\ln|u-1| + c$$
$$= 2\sqrt{x} + 2\ln|\sqrt{x} - 1| + c \checkmark$$

e $u = 2x + 3 \;\therefore\; x = \frac{1}{2}u - \frac{3}{2},\; \frac{du}{dx} = 2$

$$\int (x+1)(2x+3)^3\,dx$$
$$= \int (\tfrac{1}{2}u - \tfrac{1}{2})u^3 \times \tfrac{1}{2}\,du$$
$$= \tfrac{1}{4}\int (u^4 - u^3)\,du$$
$$= \tfrac{1}{4}(\tfrac{1}{5}u^5 - \tfrac{1}{4}u^4) + c \checkmark$$
$$= \tfrac{1}{4}[\tfrac{1}{5}(2x+3)^5 - \tfrac{1}{4}(2x+3)^4] + c \checkmark$$
$$= \tfrac{1}{80}(2x+3)^4[4(2x+3) - 5] + c$$
$$= \tfrac{1}{80}(8x+7)(2x+3)^4 + c \checkmark$$

f $u^2 = x - 2 \;\therefore\; x = u^2 + 2,\; \frac{dx}{du} = 2u$

$$\int \frac{x^2}{\sqrt{x-2}}\,dx = \int \frac{(u^2+2)^2}{u}\times 2u\,du$$
$$= 2\int (u^4 + 4u^2 + 4)\,du \checkmark$$
$$= 2(\tfrac{1}{5}u^5 + \tfrac{4}{3}u^3 + 4u) + c$$
$$= 2[\tfrac{1}{5}(x-2)^{\frac{5}{2}} + \tfrac{4}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}}] + c \checkmark$$
$$= \tfrac{2}{15}(x-2)^{\frac{1}{2}}[3(x-2)^2 + 20(x-2) + 60] + c$$
$$= \tfrac{2}{15}(3x^2 + 8x + 32)(x-2)^{\frac{1}{2}} + c \checkmark$$

2 **a** $x = \sin u \;\therefore\; \frac{dx}{du} = \cos u$

$x = 0 \Rightarrow u = 0$

$x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6}$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\ dx = \int_0^{\frac{\pi}{6}} \frac{1}{\cos u} \times \cos u\ du$$

$$= \int_0^{\frac{\pi}{6}} du$$

$$= [u]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

b $u = 2 - x \;\therefore\; x = 2 - u,\ \frac{du}{dx} = -1$

$x = 0 \Rightarrow u = 2$

$x = 2 \Rightarrow u = 0$

$$\int_0^2 x(2-x)^3\ dx = \int_2^0 (2-u)u^3 \times (-1)\ du$$

$$= \int_0^2 (2u^3 - u^4)\ du$$

$$= [\tfrac{1}{2}u^4 - \tfrac{1}{5}u^5]_0^2$$

$$= (8 - \tfrac{32}{5}) - (0)$$

$$= \frac{8}{5}$$

c $x = 2\sin u \;\therefore\; \frac{dx}{du} = 2\cos u$

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = \frac{\pi}{6}$

$$\int_0^1 \sqrt{4-x^2}\ dx$$

$$= \int_0^{\frac{\pi}{6}} 2\cos u \times 2\cos u\ du$$

$$= \int_0^{\frac{\pi}{6}} 4\cos^2 u\ du$$

$$= \int_0^{\frac{\pi}{6}} (2 + 2\cos 2u)\ du$$

$$= [2u + \sin 2u]_0^{\frac{\pi}{6}}$$

$$= (\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}) - (0)$$

$$= \tfrac{1}{6}(2\pi + 3\sqrt{3})$$

d $x = 3\tan u \;\therefore\; \frac{dx}{du} = 3\sec^2 u$

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow u = \frac{\pi}{4}$

$$\int_0^3 \frac{x^2}{x^2+9}\ dx = \int_0^{\frac{\pi}{4}} \frac{9\tan^2 u}{9\sec^2 u} \times 3\sec^2 u\ du$$

$$= 3\int_0^{\frac{\pi}{4}} \tan^2 u\ du$$

$$= 3\int_0^{\frac{\pi}{4}} (\sec^2 u - 1)\ du$$

$$= 3[\tan u - u]_0^{\frac{\pi}{4}}$$

$$= 3[(1 - \tfrac{\pi}{4}) - (0)]$$

$$= \tfrac{3}{4}(4 - \pi)$$

Q3) $u^2 = e^x - 2 \rightarrow 2u\frac{du}{dx} = e^x \rightarrow dx = \frac{2u}{e^x}\,du$ ✓

$$\int_{x=\ln 3}^{x=\ln 4} \frac{e^{4x}}{e^x - 2}\,dx = \int_{u=1}^{u=\sqrt{2}} \frac{e^{4x}}{u^2}\,\frac{2u}{e^x}\,du$$

$u^2 = 3-2 = 1$ ✓

$u^2 = 4-2 = 2$

$$= \int_{u=1}^{u=\sqrt{2}} \frac{2e^{3x}}{u}\,du \checkmark$$

$e^x = 2 + u^2$

$e^{3x} = (2+u^2)^3$ ✓

$= 8 + 12u^2 + 6u^4 + u^6$

$$= \int_1^{\sqrt{2}} \frac{16}{u} + 24u + 12u^3 + 2u^5\,du \checkmark$$

$$= \left[16\ln u + 12u^2 + 3u^4 + \tfrac{1}{3}u^6\right]_1^{\sqrt{2}} \checkmark$$

$$= \left(16\ln\sqrt{2} + 12\times 2 + 3\times 4 + \tfrac{1}{3}\times 8\right) - \left(0 + 12 + 3 + \tfrac{1}{3}\right) \checkmark$$

$$= 16\ln\sqrt{2} + 38\tfrac{2}{3} - 15\tfrac{1}{3}$$

$$= \underline{70/3 + 8\ln 2} \checkmark$$

(8)

du

Q4) let $x = \sin u \rightarrow \frac{dx}{du} = \cos u \rightarrow dx = \cos u\, du$ ✓

$x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6}$ ✓

$x = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\pi}{3}$

$\rightarrow \int_{\pi/6}^{\pi/3} \sin^2 u \underbrace{\sqrt{1-\sin^2 u}}_{= \cos u} \times \cos u \, du$

$= \int_{\pi/6}^{\pi/3} \sin^2 u \cos^2 u \, du$ ✓

$\sin 2u = 2\sin u \cos u$

$\left(\frac{1}{2}\sin 2u\right)^2 = (\sin u \cos u)^2$

$= \frac{1}{4}\int_{\pi/6}^{\pi/3} \sin^2 2u \, du$ ✓

$\sin^2 2u = \frac{1}{2} - \frac{1}{2}\cos 4u$

$= \frac{1}{8}\int_{\pi/6}^{\pi/3} 1 - \cos 4u \, du$ ✓ $= \frac{1}{8}\left[u - \frac{1}{4}\sin 4u\right]_{\pi/6}^{\pi/3}$ ✓

$= \frac{1}{8}\left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)\right]$ ✓

(8)

$= \frac{\pi}{48} + \frac{\sqrt{3}}{32}$ ✓

TOTAL

50 MARKS