Exercise 7

1) $x = t^3 \quad y = t^2$

$$\int y\,dx = \int_0^{\sqrt[3]{4}} t^2 \times 3t^2\,dt$$

x	t
0	0
4	$\sqrt[3]{4}$

$$= \int_0^{\sqrt[3]{4}} 3t^4\,dt = \left[\frac{3t^5}{5}\right]_0^{\sqrt[3]{4}} = \left(\frac{3\times(\sqrt[3]{4})^5}{5}\right) - 0$$

$$= \underline{\frac{24}{5}\sqrt[3]{2}}$$

2) $x = \sin t \quad y = \sin 2t$

t-limits: $y = 0 \Rightarrow \sin 2t = 0 \Rightarrow 2t = 0, \pi \Rightarrow t = 0, \pi/2$.

$$\int y\,dx = \int_0^{\pi/2} y\frac{dx}{dt}\,dt = \int_0^{\pi/2} \sin 2t \times \cos t\,dt$$

$$= \int_0^{\pi/2} 2\sin t\cos^2 t\,dt$$

Try $y = (\cos t)^3$
$\frac{dy}{dt} = -3\cos^2 t\sin t$

$$= \left[-\frac{2}{3}\cos^3 t\right]_0^{\pi/2}$$

$$= 0 - (-2/3) = \underline{2/3}$$

3) $x = (t+1)^2 \quad y = \frac{1}{2}t^3 + 3$

a) $\frac{dy}{dx} = \frac{3/2\,t^2}{2(t+1)} = \frac{6}{6} = 1$ when $t = 2$.

when $t = 2$, $x = (2+1)^2 = 9$, $y = \frac{1}{2}(2)^3 + 3 = 7$.

Eqn of normal: $y - 7 = -1(x - 9)$

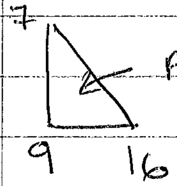
$\Rightarrow \underline{x + y = 16}$

b) $$\int_1^7 y\,dx = \int_0^2 \left(\frac{1}{2}t^3 + 3\right)(2t+2)\,dt$$

x	t
1	0
9	2

$$= \int_0^2 t^4 + t^3 + 6t + 6\,dt$$

$$= \left[\frac{t^5}{5} + \frac{t^4}{4} + \frac{6t^2}{2} + 6t\right]_0^2 = \frac{172}{5}$$



Area $= \frac{1}{2} \times 7 \times 7 = 24.5$

$24.5 + \frac{172}{5} = \underline{\frac{589}{10}}$

4) $x = 3t^2 \quad y = \sin 2t$

a) $y = 0 \Rightarrow \sin 2t = 0 \Rightarrow 2t = 0, \pi$

$t = \cancel{0}, \frac{\pi}{2}$

b) $\int_0^{\pi/2} \sin 2t \times 6t \, dt$

$u = 6t \qquad \frac{dv}{dt} = \sin 2t$

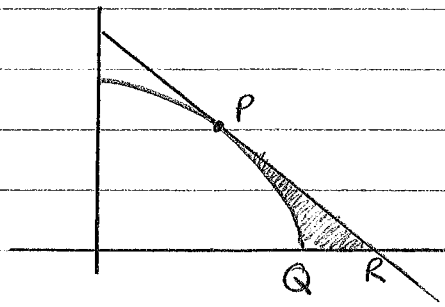
$\frac{du}{dt} = 6 \qquad v = -\frac{1}{2}\cos 2t.$

$= 6t \times -\frac{1}{2}\cos 2t - \int 6 \times -\frac{1}{2}\cos 2t \, dt.$

$= -3t\cos 2t + \int 3\cos 2t \, dt$

$= \left[-3t\cos 2t + \frac{3}{2}\sin 2t\right]_0^{\pi/2} = \left(\frac{-3\pi}{2} \times -1\right) - (0 + 0) = \frac{3\pi}{2}.$

5) $x = 5\cos\theta \quad y = 4\sin\theta$



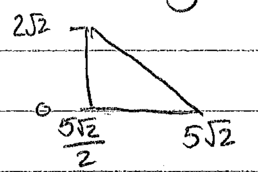
a) $\frac{dy}{dx} = \frac{4\cos\theta}{-5\sin\theta} = -\frac{4}{5}$ when $\theta = \frac{\pi}{4}.$

b) when $\theta = \pi/4$, $x = \frac{5\sqrt{2}}{2}$, $y = 2\sqrt{2}.$

$y - 2\sqrt{2} = -4/5\,(x - 5\sqrt{2}/2)$

c) Need −

At R $y = 0 \Rightarrow -2\sqrt{2} = -4/5\,(x - 5\sqrt{2}/2) \Rightarrow x = 5\sqrt{2}$



Area $= \frac{1}{2} \times 2\sqrt{2} \times \left(5\sqrt{2} - \frac{5\sqrt{2}}{2}\right) = 5.$

At Q $y = 0 \Rightarrow 4\sin\theta = 0 \quad \theta = 0,$

$x = 5\cos 0 = 5.$

$\int_{5\sqrt{2}/2}^{5} y \, dx = \int_{\pi/4}^{0} 4\sin\theta \times -5\sin\theta \, d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$20\sin^2\theta = 10 - 10\cos 2\theta$

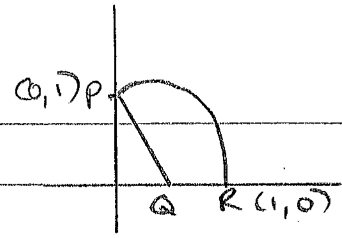
$= \int_0^{\pi/4} 20\sin^2\theta \, d\theta$

$= \int_0^{\pi/4} 10 - 10\cos 2\theta \, d\theta = \left[10\theta - 5\sin 2\theta\right]_0^{\pi/4}$

$= 5\pi/2 - 5.$

$5 - (5\pi/2 - 5) = 10 - 5\pi/2.$

− =



(6) $x = 1 - t^2 \quad y = 2t - t^3$

At P $\quad x = 0 \Rightarrow 1 - t^2 = 0$

$t = \pm 1 \qquad y = 1 \quad (t = 1)$

$\frac{dy}{dx} = \frac{2 - 3t^2}{-2t} = \frac{2-3}{-2} = \frac{1}{2}$

L has eqn $\quad y - 1 = -2(x - 0) \qquad y = -2x + 1$

R: Curve meets x-axis when $y = 0$

$\Rightarrow 2t - t^3 = 0 \Rightarrow t(2 - t^2) = 0 \quad t = 0, \pm\sqrt{2}$

when $t = 0 \quad x = 1$

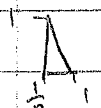
[$t = \pm\sqrt{2} \quad x = -1$ (other root)]

Area under curve

$$\int_0^1 y\,dx = \int_1^0 (2t - t^3)(-2t)\,dt$$

$$= \int_1^0 -4t^2 + 2t^4\,dt$$

$$= \int_0^1 4t^2 - 2t^4\,dt = \left[\frac{4t^3}{3} - \frac{2t^5}{5}\right]_0^1 = \frac{14}{15}.$$



$= \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{4} \qquad \frac{14}{15} - \frac{1}{4} = \frac{41}{60}.$

(At Q $\quad 0 = -2x + 1 \Rightarrow x = \frac{1}{2}$)

7) $x = t - 2\sin t \qquad y = 1 - 2\cos t$.

a) $y = 0 \Rightarrow 2\cos t = 1 \Rightarrow \cos t = \frac{1}{2} \quad t = \frac{\pi}{3}, \frac{5\pi}{3}$.

b) $$\int_{\pi/3}^{5\pi/3} (1-2\cos t)(1-2\cos t)\, dt = \int_{\pi/3}^{5\pi/3} (1-2\cos t)^2\, dt$$

c) $$= \int 1 - 4\cos t + 4\cos^2 t \, dt$$

$$= \int_{\pi/3}^{5\pi/3} 1 - 4\cos t + 2\cos 2t + 2 \, dt$$

$\cos 2t = 2\cos^2 t - 1$

$\cos 2t + 1 = 2\cos^2 t$

$$= \left[3t - 4\sin t + \frac{\cancel{2}\sin 2t}{\cancel{2}} \right]_{\pi/3}^{5\pi/3} =$$

$$= \left(\frac{\cancel{3} \times 5\pi}{\cancel{3}} - \overset{2}{\cancel{4}}\left(\frac{-\sqrt{3}}{\cancel{2}}\right) + \left(\frac{-\sqrt{3}}{2}\right) \right) - \left(\frac{\cancel{3} \times \pi}{\cancel{3}} - \overset{2}{\cancel{4}}\left(\frac{\sqrt{3}}{\cancel{2}}\right) + \left(\frac{\sqrt{3}}{2}\right) \right)$$

$$= 4\pi + 3\sqrt{3}.$$