

- 1** **a** $= 1 + \left(\frac{1}{2}\right)(-4x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(-4x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3 \times 2}(-4x)^3 + \dots$
 $= 1 - 2x - 2x^2 - 4x^3 + \dots, \quad |-4x| < 1 \quad \therefore \text{valid for } |x| < \frac{1}{4}$
- b** when $x = 0.01$, $(1 - 4x)^{\frac{1}{2}} \approx 1 - 2(0.01) - 2(0.01)^2 - 4(0.01)^3$
 $= 1 - 0.02 - 0.0002 - 0.000\ 004$
 $= 0.979\ 796$
 $(1 - 0.04)^{\frac{1}{2}} = \sqrt{0.96} = \sqrt{\frac{16 \times 6}{100}} = \frac{2}{5} \sqrt{6}$
 $\therefore \sqrt{6} \approx \frac{5}{2} \times 0.979\ 796 = 2.44949$ (6sf)
- 2** **a** $\frac{4}{1+2x-3x^2} \equiv \frac{4}{(1+3x)(1-x)} \equiv \frac{A}{1+3x} + \frac{B}{1-x}$
 $4 \equiv A(1-x) + B(1+3x)$
 $x = -\frac{1}{3} \Rightarrow 4 = \frac{4}{3}A \Rightarrow A = 3$
 $x = 1 \Rightarrow 4 = 4B \Rightarrow B = 1$
 $\therefore f(x) = \frac{3}{1+3x} + \frac{1}{1-x}$
- b** $\frac{3}{1+3x} = 3(1+3x)^{-1} = 3\left[1 + (-1)(3x) + \frac{(-1)(-2)}{2}(3x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(3x)^3 + \dots\right]$
 $= 3 - 9x + 27x^2 - 81x^3 + \dots, \quad |3x| < 1 \quad \therefore \text{valid for } |x| < \frac{1}{3}$
 $\frac{1}{1-x} = (1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(-x)^3 + \dots$
 $= 1 + x + x^2 + x^3 + \dots, \quad |-x| < 1 \quad \therefore \text{valid for } |x| < 1$
 $\therefore f(x) = (3 - 9x + 27x^2 - 81x^3 + \dots) + (1 + x + x^2 + x^3 + \dots)$
 $= 4 - 8x + 28x^2 - 80x^3 + \dots, \quad \text{valid for } |x| < \frac{1}{3}$
- 3** **a** $= 2^{-2}(1 - \frac{1}{2}x)^{-2} = \frac{1}{4}(1 - \frac{1}{2}x)^{-2}$
 $= \frac{1}{4}\left[1 + (-2)\left(-\frac{1}{2}x\right) + \frac{(-2)(-3)}{2}\left(-\frac{1}{2}x\right)^2 + \frac{(-2)(-3)(-4)}{3 \times 2}\left(-\frac{1}{2}x\right)^3 + \dots\right]$
 $= \frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \dots$
- b** $\frac{3-x}{(2-x)^2} = (3-x)(2-x)^{-2} = (3-x)\left(\frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \dots\right)$
 $\therefore \text{coefficient of } x^3 = \left(3 \times \frac{1}{8}\right) + \left(-1 \times \frac{3}{16}\right) = \frac{3}{16}$
- 4** **a** $f\left(\frac{1}{10}\right) = \frac{4}{\sqrt{1+\frac{1}{15}}} = \frac{4}{\sqrt{\frac{16}{15}}} = \frac{4}{\frac{4}{\sqrt{15}}} = 4 \times \frac{\sqrt{15}}{4} = \sqrt{15}$
- b** $= 4\left(1 + \frac{2}{3}x\right)^{-\frac{1}{2}} = 4\left[1 + \left(-\frac{1}{2}\right)\left(\frac{2}{3}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{2}{3}x\right)^2 + \dots\right]$
 $= 4 - \frac{4}{3}x + \frac{2}{3}x^2 + \dots$
- c** $\sqrt{15} = f\left(\frac{1}{10}\right) \approx 4 - \frac{4}{3} \times \frac{1}{10} + \frac{2}{3} \times \left(\frac{1}{10}\right)^2 + \dots$
 $= 4 - \frac{2}{15} + \frac{1}{150} = 3\frac{131}{150}$
- d** $\sqrt{15} = 3.872\ 98\dots$
 $3\frac{131}{150} = 3.873\ 33\dots$
 $3\frac{55}{63} = 3.873\ 01\dots$
 $\therefore \sqrt{15} < 3\frac{55}{63} < 3\frac{131}{150}$, so $3\frac{55}{63}$ is a more accurate approximation

$$5 \quad \mathbf{a} \quad = 1 + \left(\frac{1}{3}\right)(-x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}(-x)^2 + \dots$$

$$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 + \dots$$

$$\mathbf{b} \quad \text{when } x = 10^{-3}, (1 - x)^{\frac{1}{3}} \approx 1 - \frac{1}{3}(10^{-3}) - \frac{1}{9}(10^{-3})^2$$

$$= 0.999\,666\,555\,6$$

$$(1 - 10^{-3})^{\frac{1}{3}} = \sqrt[3]{0.999} = \sqrt[3]{\frac{27 \times 37}{1000}} = \frac{3}{10} \sqrt[3]{37}$$

$$\therefore \sqrt[3]{37} \approx \frac{10}{3} \times 0.999\,666\,555\,6 = 3.332\,221\,85 \text{ (9sf)}$$

$$6 \quad \mathbf{a} \quad p = \frac{\left(\frac{3}{5}\right)\left(-\frac{2}{5}\right)}{2}(5)^2 = -3$$

$$q = \frac{\left(\frac{3}{5}\right)\left(-\frac{2}{5}\right)\left(-\frac{7}{5}\right)}{3 \times 2}(5)^3 = 7$$

$$\mathbf{b} \quad \text{let } x = 0.02$$

$$(1.1)^{\frac{3}{5}} \approx 1 + 3(0.02) - 3(0.02)^2 + 7(0.02)^3$$

$$= 1 + 0.06 - 0.0012 + 0.000\,056$$

$$= 1.058\,856$$

$$\mathbf{c} \quad (1.1)^{\frac{3}{5}} = 1.058\,852\,853\dots$$

$$\% \text{ error} = \frac{1.058856 - 1.058852853}{1.058852853} \times 100\% = 0.000\,297\% \text{ (3sf)}$$

$$7 \quad \mathbf{a} \quad 8 - 6x^2 \equiv A(2+x)^2 + B(1+x)(2+x) + C(1+x)$$

$$x = -1 \quad \Rightarrow \quad A = 2$$

$$x = -2 \quad \Rightarrow \quad -16 = -C \quad \Rightarrow \quad C = 16$$

$$\text{coeffs of } x^2 \Rightarrow -6 = A + B \Rightarrow B = -8$$

$$\mathbf{b} \quad \frac{8 - 6x^2}{(1+x)(2+x)^2} \equiv \frac{2}{1+x} - \frac{8}{2+x} + \frac{16}{(2+x)^2}$$

$$\frac{2}{1+x} = 2(1+x)^{-1} = 2\left[1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3 \times 2}x^3 + \dots\right]$$

$$= 2 - 2x + 2x^2 - 2x^3 + \dots$$

$$\frac{8}{2+x} = 8(2+x)^{-1} = 8 \times 2^{-1}(1 + \frac{1}{2}x)^{-1} = 4(1 + \frac{1}{2}x)^{-1}$$

$$= 4\left[1 + (-1)\left(\frac{1}{2}x\right) + \frac{(-1)(-2)}{2}\left(\frac{1}{2}x\right)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}\left(\frac{1}{2}x\right)^3 + \dots\right]$$

$$= 4 - 2x + x^2 - \frac{1}{2}x^3 + \dots$$

$$\frac{16}{(2+x)^2} = 16(2+x)^{-2} = 16 \times 2^{-2}(1 + \frac{1}{2}x)^{-2} = 4(1 + \frac{1}{2}x)^{-2}$$

$$= 4\left[1 + (-2)\left(\frac{1}{2}x\right) + \frac{(-2)(-3)}{2}\left(\frac{1}{2}x\right)^2 + \frac{(-2)(-3)(-4)}{3 \times 2}\left(\frac{1}{2}x\right)^3 + \dots\right]$$

$$= 4 - 4x + 3x^2 - 2x^3 + \dots$$

$$\therefore \frac{8 - 6x^2}{(1+x)(2+x)^2} = (2 - 2x + 2x^2 - 2x^3 + \dots) - (4 - 2x + x^2 - \frac{1}{2}x^3 + \dots) + (4 - 4x + 3x^2 - 2x^3 + \dots)$$

$$= 2 - 4x + 4x^2 - \frac{7}{2}x^3 + \dots$$

$$8 \quad \mathbf{a} \quad = 1 + \left(\frac{1}{2}\right)(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(-2x)^2 + \dots$$

$$= 1 - x - \frac{1}{2}x^2 + \dots$$

$$\mathbf{b} \quad \text{when } x = 0.0008, (1 - 2x)^{\frac{1}{2}} \approx 1 - 0.0008 - \frac{1}{2}(0.0008)^2$$

$$= 1 - 0.0008 - 0.000\,000\,32$$

$$= 0.999\,199\,68$$

$$(1 - 0.0016)^{\frac{1}{2}} = \sqrt{0.9984} = \sqrt{\frac{256 \times 39}{10000}} = \frac{4}{25} \sqrt{39}$$

$$\therefore \sqrt{39} \approx \frac{25}{4} \times 0.999\,199\,68 = 6.244\,998 \text{ (7sf)}$$

$$9 \quad \mathbf{a} \quad = 1 + \left(\frac{1}{3}\right)(8x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}(8x)^2 + \dots$$

$$= 1 + \frac{8}{3}x - \frac{64}{9}x^2 + \dots$$

$$\mathbf{b} \quad k = \sqrt[3]{\frac{5}{1.08}} = \sqrt[3]{\frac{500}{108}} = \sqrt[3]{\frac{125}{27}} = \frac{5}{3}$$

$$\mathbf{c} \quad \text{let } x = 0.01, \sqrt[3]{1.08} = 1 + \frac{8}{3}(0.01) - \frac{64}{9}(0.01)^2$$

$$= 1.025\,955\,556$$

$$\therefore \sqrt[3]{5} = \frac{5}{3} \times 1.025\,955\,556 = 1.710 \text{ (4sf)}$$

$$10 \quad \mathbf{a} \quad f(x) \equiv \frac{6x}{(x-1)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x-3}$$

$$6x \equiv A(x-3) + B(x-1)$$

$$x = 1 \quad \Rightarrow \quad 6 = -2A \quad \Rightarrow \quad A = -3$$

$$x = 3 \quad \Rightarrow \quad 18 = 2B \quad \Rightarrow \quad B = 9$$

$$f(x) \equiv \frac{9}{x-3} - \frac{3}{x-1}$$

$$\mathbf{b} \quad f(x) = \frac{3}{1-x} - \frac{9}{3-x}$$

$$\frac{3}{1-x} = 3(1-x)^{-1} = 3\left[1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(-x)^3 + \dots\right]$$

$$= 3 + 3x + 3x^2 + 3x^3 + \dots$$

$$\frac{9}{3-x} = 9(3-x)^{-1} = 9 \times 3^{-1}(1 - \frac{1}{3}x)^{-1} = 3(1 - \frac{1}{3}x)^{-1}$$

$$= 3\left[1 + (-1)\left(-\frac{1}{3}x\right) + \frac{(-1)(-2)}{2}\left(-\frac{1}{3}x\right)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}\left(-\frac{1}{3}x\right)^3 + \dots\right]$$

$$= 3 + x + \frac{1}{3}x^2 + \frac{1}{9}x^3 + \dots$$

$$\therefore f(x) = (3 + 3x + 3x^2 + 3x^3 + \dots) - (3 + x + \frac{1}{3}x^2 + \frac{1}{9}x^3 + \dots)$$

$$= 2x + \frac{8}{3}x^2 + \frac{26}{9}x^3 + \dots$$

$$\therefore \text{for small } x, f(x) \approx 2x + \frac{8}{3}x^2 + \frac{26}{9}x^3$$

- 11 a $= 4^{\frac{1}{2}}(1 + \frac{1}{4}x)^{\frac{1}{2}} = 2(1 + \frac{1}{4}x)^{\frac{1}{2}} = 2[1 + (\frac{1}{2})(\frac{1}{4}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\frac{1}{4}x)^2 + \dots]$
 $= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \dots, |\frac{1}{4}x| < 1 \quad \therefore \text{valid for } |x| < 4$
- b when $x = \frac{1}{20}$, $(4 + x)^{\frac{1}{2}} \approx 2 + \frac{1}{4}(\frac{1}{20}) - \frac{1}{64}(\frac{1}{20})^2$
 $= 2.012460938$
 $(4 + \frac{1}{20})^{\frac{1}{2}} = \sqrt{\frac{81}{20}} = \sqrt{\frac{81 \times 5}{100}} = \frac{9}{10}\sqrt{5}$
 $\therefore \sqrt{5} \approx \frac{10}{9} \times 2.012460938 = 2.23606771$ (9sf)
- c $\sqrt{5} = 2.236067977\dots$
 \therefore estimate is accurate to 7 significant figures
- 12 a $= 1 + (-\frac{1}{2})(2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(2x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3 \times 2}(2x)^3 + \dots$
 $= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \dots$
- b $\frac{2-5x}{\sqrt{1+2x}} = (2-5x)(1+2x)^{-\frac{1}{2}} = (2-5x)(1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+\dots)$
 $= 2 - 2x + 3x^2 - 5x^3 - 5x + 5x^2 - \frac{15}{2}x^3 + \dots$
 $= 2 - 7x + 8x^2 - \frac{25}{2}x^3 + \dots$
 \therefore for small x , $\frac{2-5x}{\sqrt{1+2x}} = 2 - 7x + 8x^2 - \frac{25}{2}x^3$
- c $2 - 5x = \sqrt{3} \times \sqrt{1+2x} = \sqrt{3+6x}$
 $(2-5x)^2 = 3+6x$
 $4 - 20x + 25x^2 = 3 + 6x$
 $25x^2 - 26x + 1 = 0$
 $(25x-1)(x-1) = 0$
 $x = \frac{1}{25}, 1$
- d let $x = \frac{1}{25}$
 $\sqrt{3} \approx 2 - 7(\frac{1}{25}) + 8(\frac{1}{25})^2 - \frac{25}{2}(\frac{1}{25})^3$
 $= 1.732$
- 13 a $= 1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3 \times 2}x^3 + \dots$
 $= 1 - x + x^2 - x^3 + \dots$
- b $= 1 - bx + b^2x^2 - b^3x^3 + \dots$
- c $\frac{1+ax}{1+bx} = (1+ax)(1+bx)^{-1} = (1+ax)(1-bx+b^2x^2-b^3x^3+\dots)$
 $= 1 - bx + b^2x^2 - b^3x^3 + ax - abx^2 + ab^2x^3 + \dots$
 $= 1 + (a-b)x + (b^2-ab)x^2 + (ab^2-b^3)x^3 + \dots$
 $\therefore a - b = -4 \quad (1)$
and $b^2 - ab = 12 \quad (2)$
(1) $\Rightarrow a = b - 4$
sub. (2) $b^2 - b(b-4) = 12$
 $4b = 12$
 $b = 3, a = -1$
- d $= ab^2 - b^3 = -9 - 27 = -36$