EdExcel Statistics 1

The Normal Distribution

Section 1: Introduction to the Normal distribution

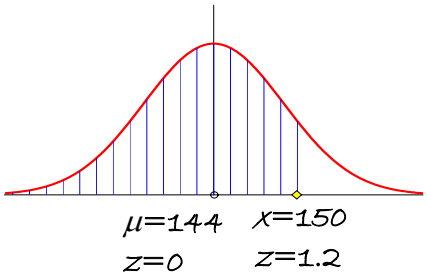
Solutions to Exercise

1. $X \sim N(144, 5^2)$

$$Z = \frac{X-144}{5} \sim N(0,1)$$

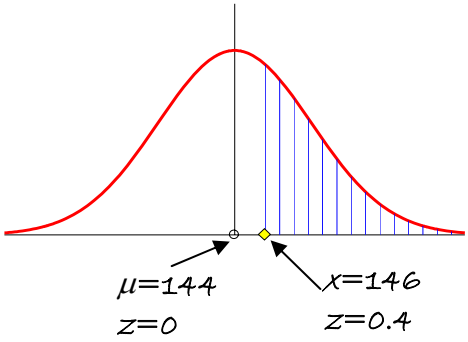
(i) $X = 150 \Rightarrow Z = \frac{150-144}{5} = 1.2$

$$\begin{aligned} P(X < 150) &= P(Z < 1.2) \\ &= \Phi(1.2) \\ &= 0.8849 \end{aligned}$$



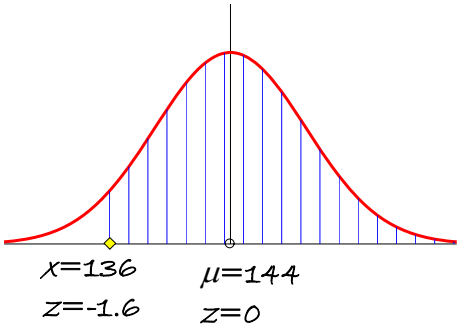
(ii) $X = 146 \Rightarrow Z = \frac{146-144}{5} = 0.4$

$$\begin{aligned} P(X > 146) &= P(Z > 0.4) \\ &= 1 - P(Z < 0.4) \\ &= 1 - \Phi(0.4) \\ &= 1 - 0.6554 \\ &= 0.3446 \end{aligned}$$



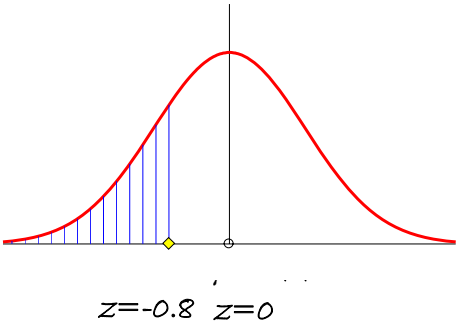
(iii) $X = 136 \Rightarrow Z = \frac{136-144}{5} = -1.6$

$$\begin{aligned} P(X > 136) &= P(Z > -1.6) \\ &= P(Z < 1.6) \\ &= \Phi(1.6) \\ &= 0.9452 \end{aligned}$$



(iv) $X = 140 \Rightarrow Z = \frac{140-144}{5} = -0.8$

$$\begin{aligned} P(X < 140) &= P(Z < -0.8) \\ &= 1 - P(Z < 0.8) \\ &= 1 - \Phi(0.8) \\ &= 1 - 0.7881 \\ &= 0.2119 \end{aligned}$$



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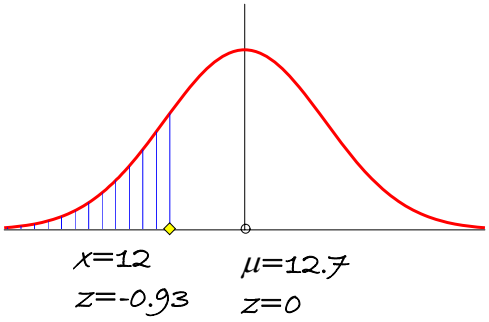
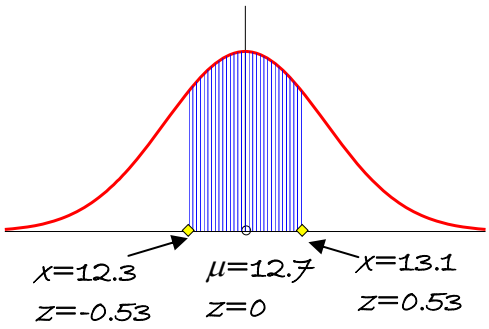
2. $X \sim N(12.7, 0.75^2)$

$$Z = \frac{X - 12.7}{0.75} \sim N(0,1)$$



(i) $X = 12 \Rightarrow Z = \frac{12 - 12.7}{0.75} = -0.93$

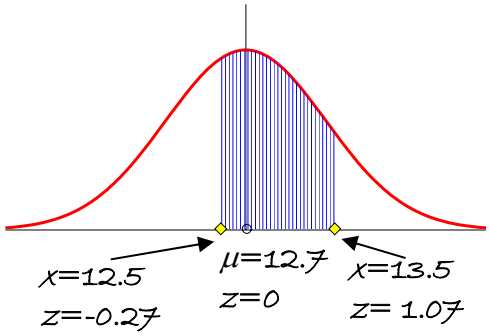
$$\begin{aligned} P(X < 12) &= P(Z < -0.93) \\ &= 1 - P(Z < 0.93) \\ &= 1 - \Phi(0.93) \\ &= 1 - 0.8238 \\ &= 0.1762 \end{aligned}$$



(ii) $X = 13.1 \Rightarrow Z = \frac{13.1 - 12.7}{0.75} = 0.53$

$X = 12.3 \Rightarrow Z = \frac{12.3 - 12.7}{0.75} = -0.53$

$$\begin{aligned} P(12.3 < X < 13.1) &= P(-0.53 < Z < 0.53) \\ &= \Phi(0.53) - \Phi(-0.53) \\ &= \Phi(0.53) - (1 - \Phi(-0.53)) \\ &= 0.7019 - 1 + 0.7019 \\ &= 0.4038 \end{aligned}$$



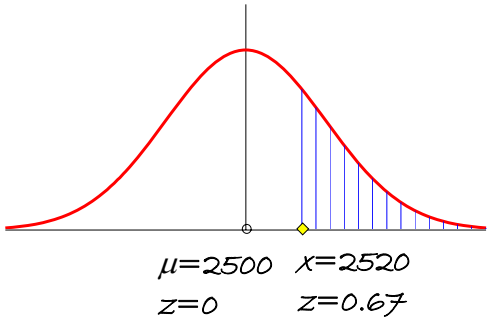
(iii) $X = 13.5 \Rightarrow Z = \frac{13.5 - 12.7}{0.75} = 1.07$

$X = 12.5 \Rightarrow Z = \frac{12.5 - 12.7}{0.75} = -0.27$

$$\begin{aligned} P(12.5 < X < 13.5) &= P(-0.27 < Z < 1.07) \\ &= \Phi(1.07) - \Phi(-0.27) \\ &= \Phi(1.07) - (1 - \Phi(0.27)) \\ &= 0.8577 - 1 + 0.6064 \\ &= 0.4641 \end{aligned}$$

3. $X \sim N(2500, 30^2)$

$$Z = \frac{X - 2500}{30} \sim N(0,1)$$

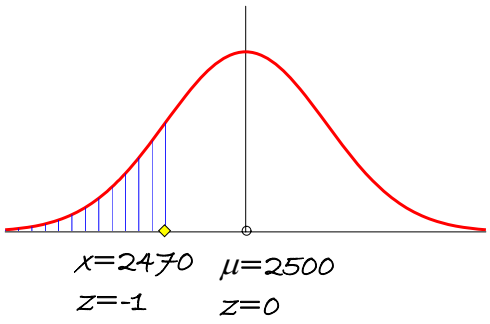


(i) (a) $X = 2520 \Rightarrow Z = \frac{2520 - 2500}{30} = 0.67$

$$P(X > 2520) = P(Z > 0.67)$$
$$= 1 - P(Z < 0.67)$$
$$= 1 - \Phi(0.67)$$
$$= 1 - 0.7486$$
$$= 0.2514$$

(b) $X = 2470 \Rightarrow Z = \dfrac{2470 - 2500}{30} = -1$

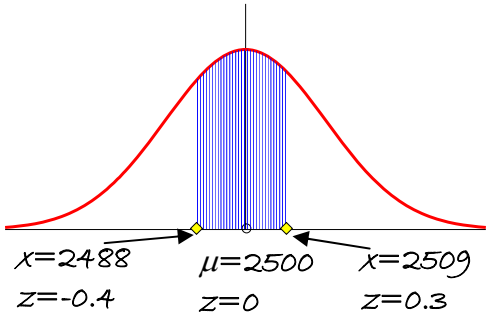
$$P(X < 2470) = P(Z < -1)$$
$$= 1 - P(Z < 1)$$
$$= 1 - \Phi(1)$$
$$= 1 - 0.8413$$
$$= 0.1587$$



(c) $X = 2488 \Rightarrow Z = \dfrac{2488 - 2500}{30} = -0.4$

$X = 2509 \Rightarrow Z = \dfrac{2509 - 2500}{30} = 0.3$

$$P(2488 < X < 2509) = P(-0.4 < Z < 0.3)$$
$$= \Phi(0.3) - \Phi(-0.4)$$
$$= \Phi(0.3) - (1 - \Phi(0.4))$$
$$= 0.6179 - 1 + 0.6554$$
$$= 0.2733$$



(ii) (a) Need a such that 35% of the components have lifetimes greater than a.

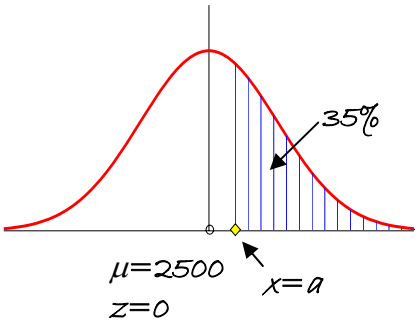
$P(X > a) = 0.35$

$P(X < a) = 0.65$

$\Phi\left(\dfrac{a - 2500}{30}\right) = 0.65$

$\dfrac{a - 2500}{30} = 0.39$

$a = 2512$ hours



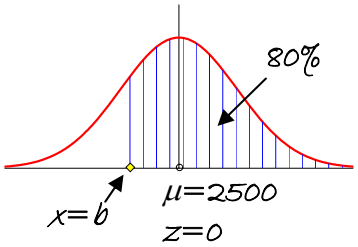
(b) 50% of the components have lifetimes greater than the mean.
So 50% of the components would last 2500 hours.

(c) Need b such that 80% of the components have lifetimes greater than b.

$P(X > b) = 0.8$

$P(X < b) = 0.2$

$\Phi\left(\dfrac{b - 2500}{30}\right) = 0.2$



$$\Phi\left(\frac{2500-b}{30}\right)=0.8$$

$$\frac{2500-b}{30}=0.84$$

$b = 2475$ hours

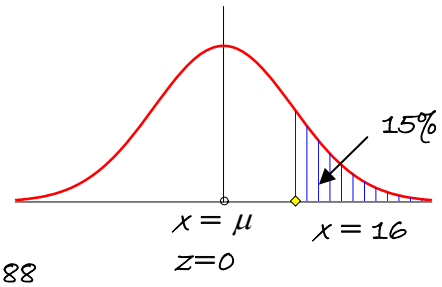
4. $X \sim N(\mu, 3^2)$

$$Z=\frac{X-\mu}{3}\sim N(0,1)$$

$P(X>16)=0.15$

$P(X<16)=0.85$

$$\Phi\left(\frac{16-\mu}{3}\right)=0.85 \quad \Rightarrow \frac{16-\mu}{3}=1.04 \quad \Rightarrow \mu=12.88$$

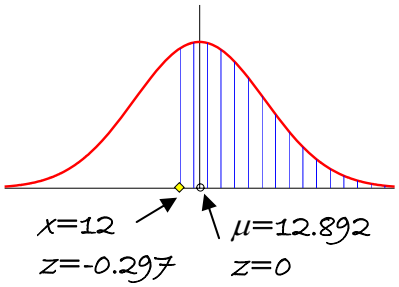


$$X=12 \quad \Rightarrow Z=\frac{12-12.88}{3}=-0.29$$

$P(X>12) = P(Z>-0.29)$

$= P(Z<0.29)$

$= 0.6141$

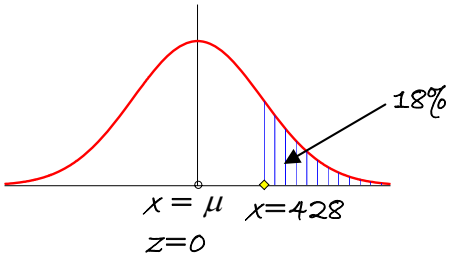


5. $$Z=\frac{X-\mu}{\sigma}\sim N(0,1)$$

$P(X>428)=0.18$

$P(X<428)=0.82$

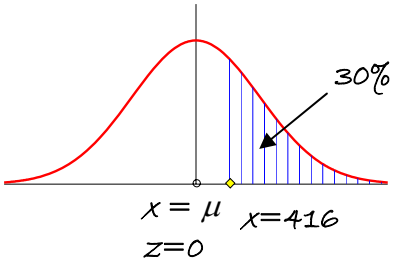
$$\Phi\left(\frac{428-\mu}{\sigma}\right)=0.82 \quad \Rightarrow \frac{428-\mu}{\sigma}=0.92$$



$P(X>416)=0.3$

$P(X<416)=0.7$

$$\Phi\left(\frac{416-\mu}{\sigma}\right)=0.7 \quad \Rightarrow \frac{416-\mu}{\sigma}=0.52$$



$$428-\mu=0.92\sigma$$

$$416-\mu=0.52\sigma$$

Subtracting: $12=0.4\sigma$

$\sigma = 30$

$428-\mu=0.92\times30 \quad \Rightarrow \mu=400.4$

The mean is 400 g and the standard deviation is 30 g.

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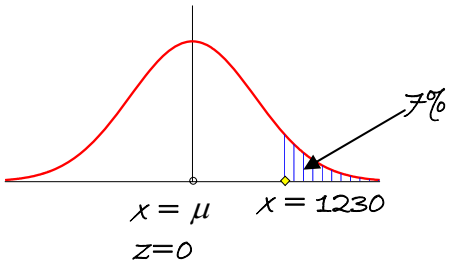
6. $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

$P(X > 1230) = 0.07$

$P(X < 1230) = 0.93$

$\Phi\left(\dfrac{1230-\mu}{\sigma}\right) = 0.93$

$\dfrac{1230-\mu}{\sigma} = 1.48$

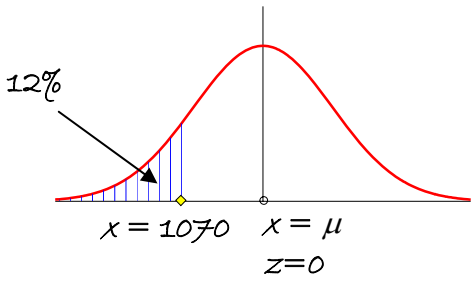


$P(X < 1070) = 0.12$

$\Phi\left(\dfrac{1070-\mu}{\sigma}\right) = 0.12$

$\Phi\left(\dfrac{\mu-1070}{\sigma}\right) = 0.88$

$\dfrac{\mu-1070}{\sigma} = 1.175$



$1230 - \mu = 1.48\sigma$

$\mu - 1070 = 1.175\sigma$

Adding: $160 = 2.655\sigma$

$\sigma = 60.3$

$1230 - \mu = 1.48 \times 60.3 \quad \Rightarrow \mu = 1141$

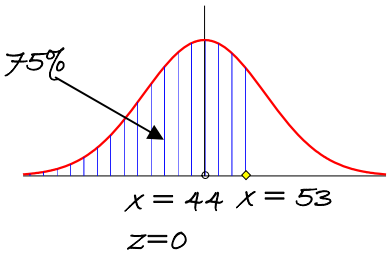
The mean is 1141 hours and the standard deviation is 60.3 hours.

7. $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

By symmetry mean $= \dfrac{35+53}{2} = 44$

$P(X < 53) = 0.75$

$\Phi\left(\dfrac{53-44}{\sigma}\right) = 0.75 \quad \Rightarrow \dfrac{9}{\sigma} = 0.67 \quad \Rightarrow \sigma = 13.4$

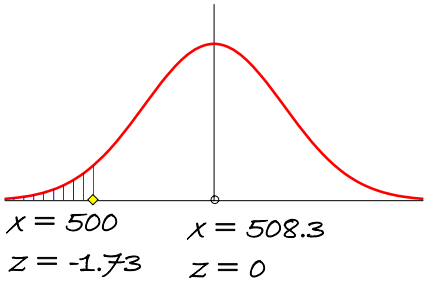


The mean is 44 and the variance is 180.4.

8. $X \sim N(508.3, 4.8^2)$

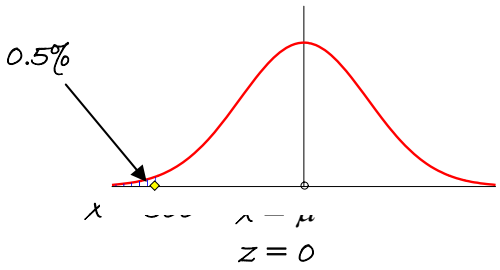
$Z = \dfrac{X-508.3}{4.8} \sim N(0,1)$

$X = 500 \quad \Rightarrow Z = \dfrac{500-508.3}{4.8} = -1.73$



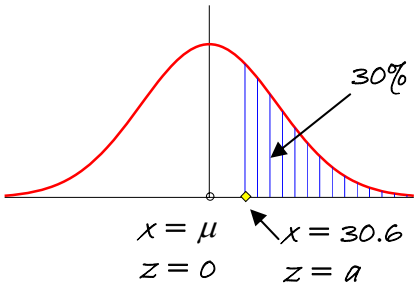
$$P(X < 500) = P(Z < -1.73)$$
$$= \Phi(-1.73)$$
$$= 1 - \Phi(1.73)$$
$$= 1 - 0.9582$$
$$= 0.0418$$

$$P(X < 500) = 0.005$$
$$\Phi\left(\frac{500-\mu}{4.8}\right) = 0.005$$
$$\Phi\left(\frac{\mu-500}{4.8}\right) = 0.995$$
$$\frac{\mu-500}{4.8} = 2.6$$
$$\mu = 512.5$$

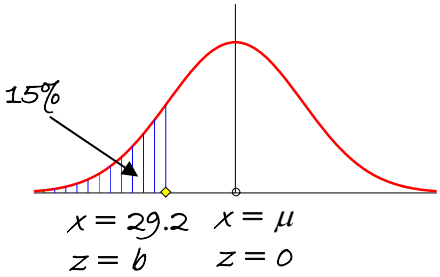


9. (i) $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$$P(X > 30.6) = 0.3$$
$$P(X < 30.6 = 0.7$$
$$\Phi\left(\frac{30.6-\mu}{\sigma}\right) = 0.7$$
$$\frac{30.6-\mu}{\sigma} = 0.52$$



$$P(X < 29.2) = 0.15$$
$$\Phi\left(\frac{29.2-\mu}{\sigma}\right) = 0.15$$
$$\Phi\left(\frac{\mu-29.2}{\sigma}\right) = 0.85$$
$$\frac{\mu-29.2}{\sigma} = 1.04$$



$$30.6 - \mu = 0.52\sigma$$
$$\mu - 29.2 = 1.04\sigma$$

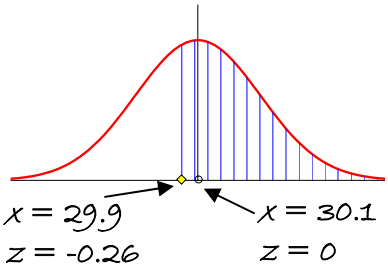
Adding: $1.4 = 1.56\sigma$

$$\sigma = 0.897$$

$30.6 - \mu = 0.52 \times 0.897 \Rightarrow \mu = 30.1$

The mean is 30.1 cm and the standard deviation is 0.897 cm.

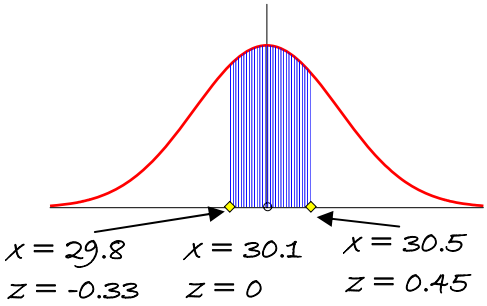
(ii) $X = 29.9 \Rightarrow Z = \frac{29.9-30.1}{0.897} = -0.26$



$$P(X > 29.9) = P(Z > -0.26)$$
$$= 1 - P(Z < -0.26)$$
$$= 1 - \Phi(-0.26)$$
$$= \Phi(0.26)$$
$$= 0.6026$$

(iii) $X = 29.8 \Rightarrow Z = \dfrac{29.8 - 30.1}{0.897} = -0.33$

$$P(X < 29.8) = P(Z < -0.33)$$
$$= 1 - P(Z < 0.33)$$
$$= 1 - \Phi(0.33)$$
$$= 1 - 0.6293$$
$$= 0.3707$$



$$X = 30.5 \Rightarrow Z = \frac{30.5 - 30.1}{0.897} = 0.45$$
$$P(X > 30.5) = P(Z > 0.45)$$
$$= 1 - P(Z < 0.45)$$
$$= 1 - \Phi(0.45)$$
$$= 1 - 0.6736$$
$$= 0.3264$$

Percentage rejected = 37.07 + 32.64 = 69.7%