Chapter 1 Rotational dynamics

1.1 Angular acceleration

Learning objectives

→ Explain what is meant by angular acceleration.

→ Calculate the angular acceleration of a rotating object when it speeds up and slows down.

→ Calculate the number of turns a rotating object makes in a particular time when it accelerates uniformly.

Angular speed and angular acceleration

There are many examples of rotating objects around you. Examples include the Earth spinning on its axis, the wheels of a moving car, a flywheel in motion, and the rotation of the armature coil of an electric motor.

When a rigid body rotates about a fixed axis:

- The angle it turns through is called its **angular displacement**, $\Delta\theta$. The unit of angular displacement is the radian (abbreviated as rad), where 2π radians = 360°.
- For angular displacement $\Delta\theta$, the number of turns made, $n = \dfrac{\Delta\theta}{2\pi}$. For example,

  if $\Delta\theta = 13.4$ radians, $n = 2.13$ turns (to three significant figures) $\left(= \dfrac{13.4}{2\pi}\right)$.
- Its **angular speed**, ω, is the change in its angular displacement per second. The unit of angular speed is the radian per second (i.e., rad s^{-1}). Therefore, for a rotating object that turns through angular displacement, $\Delta\theta$, in a time interval Δt:

$$\textbf{its average angular speed } \omega = \frac{\Delta\theta}{\Delta t}$$

If the object is rotating at constant angular speed, ω, its angular displacement $\Delta\theta$ in one rotation is 2π radians. Therefore:

$$\omega = \frac{2\pi}{T}$$

where T is its period of rotation.

- If the angular speed of the rotating body changes, then its **angular acceleration**, α, is the change in its angular velocity per second. Therefore:

$$\textbf{its angular acceleration } \alpha = \frac{\Delta\omega}{\Delta t}$$

The unit of angular acceleration is the radian per second per second (i.e., rad s^{-2}).

For constant angular acceleration, the change in angular speed is $\Delta\omega$ in a time interval Δt.

Notes

1 Angular speed is sometimes expressed in revolutions per minute or rpm. To convert to rad s^{-1}, multiply by $\frac{2\pi}{60}$ as there are 2π radians per revolution and 60 seconds per minute.

2 Angular velocity is angular speed in a certain direction of rotation.

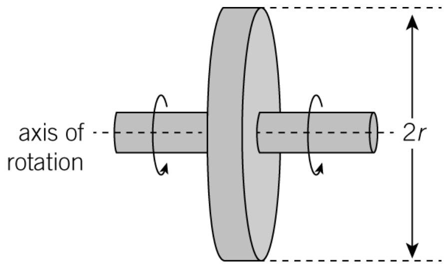

Figure 1 *A flywheel in motion*

Link

Angular speed was looked at in Topic 17.1, Uniform circular motion, in Year 2 of the *AQA Physics* student book.

When a flywheel rotates at constant angular speed, its angular acceleration is zero because its angular speed does not change. If ω is its angular speed, its period of rotation $T = \frac{2\pi}{\omega}$. If the flywheel has a radius r, a point on the flywheel rim moves on a circular path at a speed $v = \omega r$.

When a flywheel speeds up, every point of the flywheel moves at increasing speed. If the flywheel speeds up from initial angular speed ω_1 to angular speed ω_2 in time t, then:

- its angular acceleration $\alpha = \left(\frac{\omega_2 - \omega_1}{t}\right)$
- the speed of any point on its rim increases from speed $u = \omega_1 r$ to speed $v = \omega_2 r$ in time t, where r is the radius of the flywheel.

The speeding-up process accelerates such a point along a circular path of radius r. Therefore the acceleration, a, of the point along its path may be calculated from:

$$a = \left(\frac{v - u}{t}\right) = \frac{\omega_2 r - \omega_1 r}{t} = \left(\frac{\omega_2 - \omega_1}{t}\right) r = \alpha r$$

In general, for any rotating body, for any point (small 'element') in the body at perpendicular distance r from the axis of rotation,

- its speed $v = \omega r$
- its acceleration $a = \alpha r$

Every part of a rotating object experiences the same angular acceleration. However, the acceleration, a, at a point in the body is along its circular path (i.e., its linear acceleration) and is proportional to r (because $a = \alpha r$).

Note

Every point of a rotating object experiences a centripetal acceleration equal to $\omega^2 r$ which acts directly towards the centre of rotation of that point.

Worked example

A flywheel is speeded up from 5.0 to 11.0 revolutions per minute in 100s. The radius of the flywheel is 0.080 m. Calculate:

a the angular acceleration of the flywheel

b the acceleration of a point on the rim along its circular path.

Solution

a Initial angular speed, $\omega_1 = 5.0 \times \dfrac{2\pi}{60} = 0.52\ \text{rad s}^{-1}$

angular speed after 100 s, $\omega_2 = 11.0 \times \dfrac{2\pi}{60} = 1.15\ \text{rad s}^{-1}$

angular acceleration, $\alpha = \left(\dfrac{\omega_2 - \omega_1}{t}\right) = \dfrac{1.15 - 0.52}{100} = 6.3 \times 10^{-3}\ \text{rad s}^{-2}$

b Acceleration tangential to the rim, $a = \alpha r = 6.3 \times 10^{-3} \times 0.080 = 5.0 \times 10^{-4}\ \text{m s}^{-1}$

Equations for constant angular acceleration

These may be derived in much the same way as the equations for straight line motion with constant acceleration.

1 From the definition of angular acceleration α above, you can obtain by rearrangement,

$$\omega_2 = \omega_1 + \alpha t$$

where ω_1 = initial angular speed, and ω_2 = angular speed at time t.

2 The average value of the angular speed is obtained by averaging the initial and final values, giving an average value of $\left(\dfrac{\omega_1 + \omega_2}{2}\right)$.

The angle which the object turns through, the angular displacement θ, is equal to the average angular speed × the time taken which gives

$$\theta = \left(\frac{\omega_1 + \omega_2}{2}\right)t$$

3 The two equations above can be combined to eliminate ω_2 to give

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

4 Alternatively, the first two equations may be combined to eliminate *t*, giving

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta$$

Note

The four equations above are directly comparable with the four linear dynamics equations. The task of using the angular equations is made much easier by translating between linear and rotational terms, as shown in Table 1.

Table 1

Linear equations		Rotational equations
displacement s	$\longleftrightarrow$	angular displacement θ
speed or velocity u v	 $\longleftrightarrow$ $\longleftrightarrow$	angular speed ω_1 ω_2
acceleration a	$\longleftrightarrow$	angular acceleration α
For example		
$v = u + at$	$\longleftrightarrow$	$\omega_2 = \omega_1 + \alpha t$
$s = \left(\frac{u+v}{2}\right)t$	$\longleftrightarrow$	$\theta = \left(\frac{\omega_1+\omega_2}{2}\right)t$
$s = ut + \frac{at^2}{2}$	$\longleftrightarrow$	$\theta = \omega_1 t + \frac{\alpha t^2}{2}$
$v^2 = u^2 + 2as$	$\longleftrightarrow$	$\omega_2^2 = \omega_1^2 + 2\alpha\theta$

Link

The linear dynamics equations for constant acceleration were looked at in Topic 7.3, Motion along a straight line at constant acceleration, in Year 1 of the *AQA Physics* student book.

Rotational motion graphs

Rotational motion can be represented graphically in a similar way to linear motion as shown in Table 2.

Table 2

Rotational motion graph	Equivalent linear motion graph
1. Angular displacement θ against time *t*	**1. Displacement *s* against time *t***
gradient = angular velocity ω	gradient = velocity *v*
2. Angular velocity ω against time *t*	**2. Velocity *v* against time *t***
gradient = angular acceleration α	gradient = (linear) acceleration *a*
area under the graph line = angular displacement θ	area under the graph line = displacement *s*

The angular displacement graph in Figure 2 shows how the angular displacement of a rotating object changes with time. In this example, the gradient of the line increases from zero then becomes constant then decreases to zero. If the graph showed how the displacement of an object varied with time, you would be able to tell from its gradient that the velocity increased from zero and reached a constant value then decreased to zero again. The gradient of the line in Figure 2 tells you that the object's angular velocity increases from zero to a constant value then decreases to zero. The angular velocity at any point on the line is given by the gradient of the line at that point.

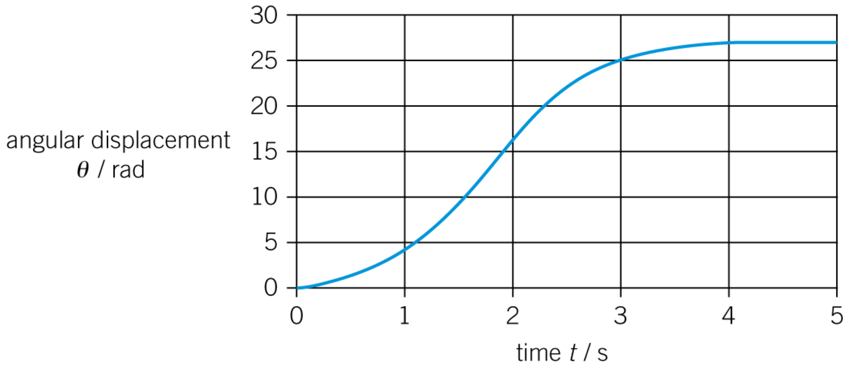

Figure 2 *Graph of angular displacement plotted against time*

The angular velocity graph in Figure 3 shows how the angular velocity of a different rotating object changes with time. In this example, the angular velocity increases to a peak then decreases to zero.

- The gradient of the line gives the angular acceleration of the object. The gradient is constant at first, then it decreases and then becomes negative. So the gradient tells you that the object has a constant angular acceleration at first. Then its angular acceleration decreases to zero when its angular velocity is a maximum. Then its angular acceleration becomes negative, so its angular velocity decreases until it stops rotating.
- The area under the line gives the angular displacement of the object. The area under the last part of the line where the angular velocity decreases is greater than the area under the first part of the line where the angular velocity increases. This tells you that the angular displacement during the last part is greater than during the first part. In other words, the object makes more turns as it slows down than it does as it speeds up.

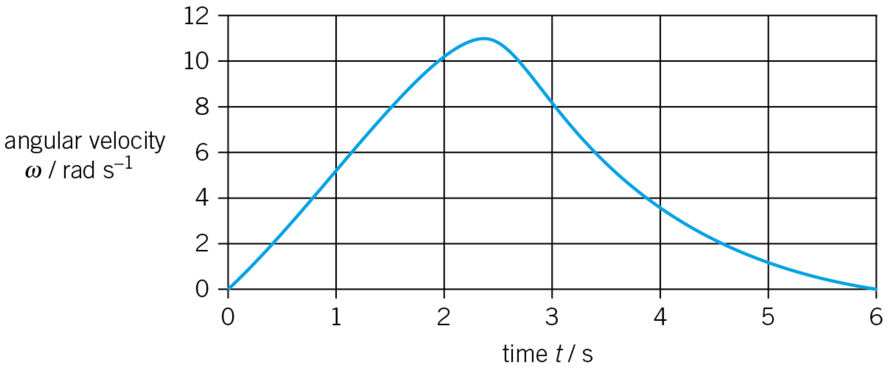

Figure 3 *Graph of angular velocity plotted against time*

Summary questions

1 A flywheel accelerates uniformly from rest to 12 rad s^{-1} in 60 seconds. Calculate:

- **a** its angular acceleration
- **b** **i** the angle it turned through in this time
  - **ii** the number of turns it made.

2 A child's spinning top, spinning at a frequency of 12 Hz, decelerated uniformly to rest in 50 s. Calculate:

- **a** its initial angular speed
- **b** its angular deceleration
- **c** the number of turns it made when it decelerated.

3 A spin-drier tub accelerated uniformly from rest to an angular speed of 1100 revolutions per second in 50 seconds. Calculate:

- **a** the angle which the tub turned through in this time
- **b** the number of turns it made as it accelerated.

4 A vehicle with wheels of diameter 0.45 m decelerated uniformly from a speed of 24 m s^{-1} to a standstill in a distance of 60 m. Calculate:

- **a** the angular speed of each wheel when the vehicle was moving at 24 m s^{-1}
- **b** **i** the time taken by the vehicle to decelerate to a standstill from a speed of 24 m s^{-1}
  - **ii** the number of turns each wheel made during the time the vehicle slowed down.
- **c** the angular speed of the vehicle one second before it stopped.

5 **a** Figure 2 shows how the angular displacement of an object changes with time. Determine the maximum angular velocity of the object.

- **b** Figure 3 shows how the angular velocity of an object changes with time.
  - **i** Determine the maximum angular acceleration of the object.
  - **ii** Estimate the total angular displacement of the object and hence determine how many turns the object makes.

1.2 Moment of inertia

Learning objectives

→ Define torque.

→ Explain what is meant by moment of inertia.

→ Describe how the angular acceleration of a rotating object depends on its moment of inertia.

Torque

To make a flywheel rotate, a turning force must be applied to it. The turning effect depends not just on the force but also on where it is applied. The torque of a turning force is the moment of the force about the axis. Therefore, torque is defined as follows.

Torque = force × perpendicular distance from the axis to the line of action of the force

The unit of torque is the newton metre (N m).

The above definition gives the equation

$$\textbf{torque } T = Fr$$

where the force F that causes the torque acts at perpendicular distance r from the axis to the line of action of the force.

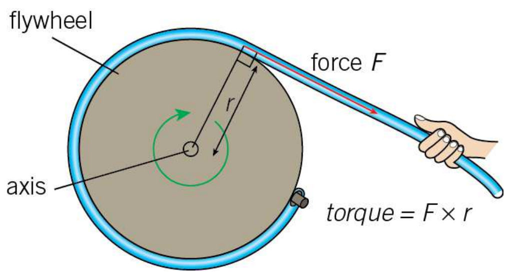

Figure 1 *Applying a torque*

Link

The moment of a force was looked at in Topic 6.3, The principle of moments, in Year 1 of the *AQA Physics* student book.

The **inertia** of an object is its resistance to change of its motion. If a large torque is required to start a flywheel turning, the flywheel must have considerable inertia. In other words, its resistance to change of its motion is high.

Every object has the property of inertia because every object has mass. However, the inertia of a rotating body depends on the distribution of its mass about the axis of rotation as well as the amount of mass. For example, the moment of inertia of a rod about an axis perpendicular to the rod is very different if the axis is through one end of the rod than if it is through the centre of the rod.

Consider a flat rigid body which can be rotated about an axis perpendicular to its plane, as shown in Figure 2. Suppose it is initially at rest and a torque is applied to it to make it rotate. Assuming there is no friction on its bearing, the applied torque will increase its angular speed. When the torque is removed, the angular speed stops increasing so it turns at constant frequency once the torque is removed.

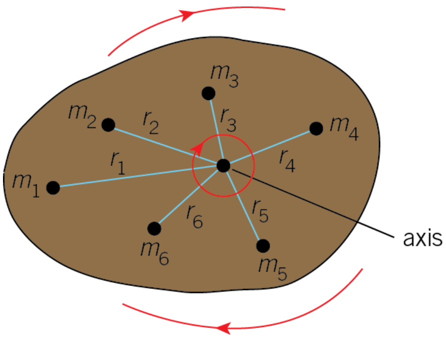

Figure 2 *A rigid body considered as a network of point masses*

The body in Figure 2 may be thought of as a network of point masses, m_1, m_2, m_3, etc., at perpendicular distances r_1, r_2, r_3, etc. from the axis. Each point turns on a circular path about the axis.

At angular speed ω, the speed of each point is given by $v = \omega r$ so:

- the speed of m_1 is ωr_1
- the speed of m_2 is ωr_2, etc.

When the body speeds up, every point in it accelerates. If the angular acceleration of the body is α, then the acceleration of each point mass is given by $a = \alpha r$.

So:

- the acceleration of m_1 is αr_1
- the acceleration of m_2 is αr_2, etc.

Using $F = ma$, the force needed to accelerate each point mass is therefore given by:

- $F_1 = m_1 \alpha r_1$ for m_1
- $F_2 = m_2 \alpha r_2$ for m_2, etc.

The moment needed for each point mass to be given angular acceleration α is given by force $F \times$ distance, r, since moment = force $\times$ perpendicular distance from the point mass to the axis. Thus:

- the moment for $m_1 = (m_1 \alpha r_1)r_1$
- the moment for $m_2 = (m_2 \alpha r_2)r_2$, etc.

The total moment (torque T) needed to give the body angular acceleration α = the sum of the individual moments needed for all the point masses.

Hence torque $T = (m_1 r_1^2)\alpha + (m_2 r_2^2)\alpha + (m_3 r_3^2)\alpha + \ldots\ldots\ldots + (m_N r_N^2)\alpha$

$$= [(m_1 r_1^2) + (m_2 r_2^2) + (m_3 r_3^2) + \ldots\ldots\ldots + (m_N r_N^2)]\alpha$$

$$= I\alpha$$

where $I = [(m_1 r_1^2) + (m_2 r_2^2) + (m_3 r_3^2) + \ldots\ldots\ldots + (m_N r_N^2)]$ is the **moment of inertia** of the body about the axis of rotation.

The summation $(m_1r_1^2) + (m_2r_2^2) + (m_3r_3^2) + \ldots + (m_Nr_N^2)$ is written in 'short form' as Σmr^2 (pronounced 'sigma m r squared').

The moment of inertia I of a body about a given axis is defined as $\Sigma m_i r_i^2$ (or as in the specification Σmr^2) **for all the points in the body, where m_i is the mass of each point and r_i is its perpendicular distance from the axis.**

The unit of I is kg m^2.

In general, T is the resultant torque. For example, if a torque T_1 is applied to a flywheel which is also acted on by a frictional torque T_2, the resultant torque is $T_1 - T_2$.

Therefore when a body undergoes angular acceleration α, the resultant torque T acting on it is given by:

$$T = I\alpha$$

Study tip

The derivation of $I = \Sigma mr^2$ is not required in this specification.

Worked example

A flywheel of moment of inertia 0.45 kg m^2 is accelerated uniformly from rest to an angular speed of 6.7 rad s^{-1} in 4.8 s. Calculate the resultant torque acting on the flywheel during this time.

Solution

Angular acceleration $\alpha = \dfrac{(\omega_2 - \omega_1)}{t} = \dfrac{6.7 - 0}{4.8} = 1.4 \text{ rad s}^{-2}$

Resultant torque $T = I\alpha = 0.45 \times 1.4 = 0.63$ N m

Moment of inertia and angular acceleration

When a resultant torque is applied to a body, the angular acceleration, α, of the body is given by $\alpha = \dfrac{T}{I}$. Thus the angular acceleration depends not just on the torque, T, but also on the moment of inertia, I, of the body about the given axis which is determined by the distribution of mass about the axis.

Two bodies of equal mass distributed in different ways will have different values of I. For example, compare a hoop and a disc of the same mass, as in Figure 3.

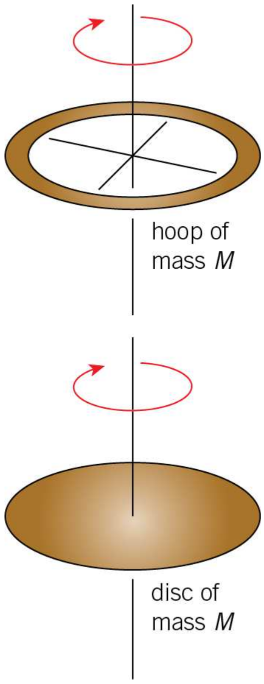

Figure 3 *Distribution of mass*

About the axis shown:

- the moment of inertia of the hoop is simply MR^2 where M is its mass and R is its radius. This is because all the mass of the hoop is at the same distance ($= R$) from the axis. The moment of inertia I about the axis shown, Σmr^2, is therefore just MR^2 for the hoop.
- the moment of inertia of the disc about the same axis is less than MR^2 because the mass of the disc is distributed between the centre and the rim. Detailed theory shows that the value of I for the disc about the axis shown is $\frac{1}{2}MR^2$.

In general,

**the further the mass is distributed from the axis,
the greater is the moment of inertia about that axis.**

The moment of inertia about a given axis of an object with a simple geometrical shape can be calculated using an appropriate mathematical equation for that shape and axis. In general terms, such equations include geometrical factors as well as mass. For example:

- the hoop as shown in Figure 3 has a moment of inertia given by MR^2
- the disc shown in Figure 3 has a moment of inertia given by $\frac{1}{2}MR^2$
- a uniform beam of length, L, and mass, M, has a moment of inertia about an axis perpendicular to its length given by:
  - $\frac{ML^2}{12}$ if the axis is through its centre
  - $\frac{ML^2}{3}$ if the axis is through its end.

The equations above and for other bodies with simple shapes are derived using the mathematical technique of integration to obtain Σmr^2 for all points in the body. The emphasis in physics and engineering is on the use rather than the derivation of the equations. By using the appropriate equation, the moment of inertia of a simple shape can be calculated from its dimensions and mass. Then, when subjected to torque, the effect on the motion can be determined. Apart from a circular hoop, you are not expected to recall the equation for the moment of inertia of any object and questions will supply, if necessary, any such equation.

Worked example

A solid circular drum of mass 4.0 kg and radius 0.15 m is rotating at an angular speed of 22 rad s^{-1} about an axis as shown in Figure 4 when a 'braking' torque is applied to it which brings it to rest in 5.8 s.

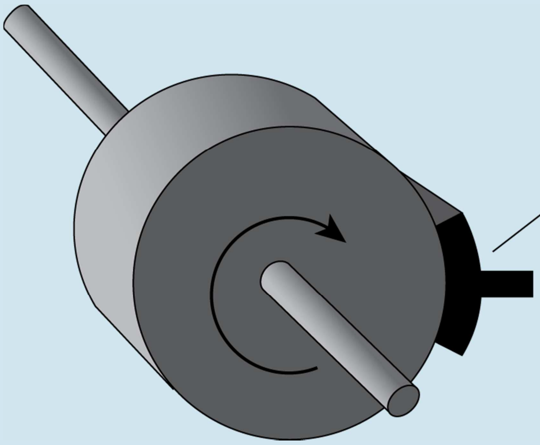

Figure 4

Calculate:

a its angular deceleration when the braking torque is applied

b the moment of inertia of the drum about the axis shown

c the resultant torque that causes it to decelerate.

Moment of inertia of drum about the axis shown = $\frac{1}{2}MR^2$

Solution

a Angular acceleration $\alpha = \frac{(\omega_2 - \omega_1)}{t} = \frac{0-22}{5.8} = -3.8 \text{ rad s}^{-2}$

b $I = \frac{1}{2}MR^2 = 0.5 \times 4.0 \times 0.15^2 = 0.045 \text{ kg m}^2$

c Resultant torque $T = I\alpha = 0.045 \times 3.8 = 0.17$ N m

Summary questions

1 The rotating part of an electric fan has a moment of inertia of 0.68 kg m^2. The rotating part is accelerated uniformly from rest to an angular speed of 3.7 rad s^{-1} in 9.2 s. Calculate the resultant torque acting on the fan during this time.

2 A solid circular disc of mass 7.4 kg and radius 0.090 m is mounted on an axis as in Figure 1. A force of 7.0 N is applied tangentially to the disc at its rim as shown in Figure 1 to accelerate the disc from rest.

a Show that:

i the moment of inertia of the disc about this axis is 0.030 kg m^2

ii the torque applied to the disc is 0.63 N m.

b The force of 7.0 N is applied for 15.0 s. Calculate:

i the angular acceleration of the disc at the end of this time

ii the number of turns made by the disc in this time.

Moment of inertia of disc about the axis shown $= \frac{1}{2}MR^2$

3 Figure 5 shows cross-sections of two discs X and Y which have the same mass and radius. State and explain which disc has the greater moment of inertia about the axis shown.

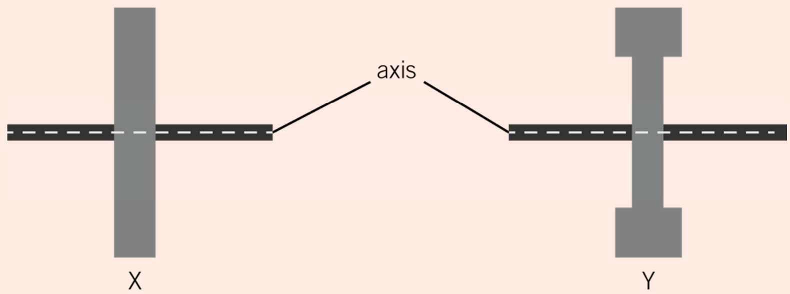

Figure 5

4 A flywheel is accelerated by a constant torque for 18 s from rest. During this time it makes 36 turns. It then slows down to a standstill 92 s after the torque is removed, making 87 turns during this time.

a **i** Show that in the time it accelerated, its angular acceleration was 1.40 rad s^{-2}.

ii Show that in the time it slowed down, its angular deceleration was 0.13 rad s^{-2}.

b The torque applied to it when it accelerated was 26 N m.

i Show that the frictional torque that slowed it down was 2.2 N m.

ii Calculate the moment of inertia of the flywheel.

1.3 Rotational kinetic energy

Learning objectives

→ State what the kinetic energy of a rotating object depends on.
→ Explain how much work a torque does when it makes a rotating object turn.
→ Measure the moment of inertia of a flywheel.

Kinetic energy

To make a body which is initially at rest rotate about a fixed axis, it is necessary to apply a torque to the body. The torque does work on the body and, as long as the applied torque exceeds the frictional torque, the work done increases the kinetic energy of the body and the faster the body rotates.

The kinetic energy, E_K, of a body rotating at angular speed ω is given by

$$\boldsymbol{E_K = \frac{1}{2}I\omega^2}$$

where I is its moment of inertia about the axis of rotation.

To prove this equation, consider the body as a network of point masses m_1, m_2, m_3, m_4, etc. When the body rotates at angular speed ω, the speed of each point mass is given by $v = \omega r$ (see Figure 2 in Topic 1.2).

- speed of $m_1 = \omega r_1$, where r_1 is the distance of m_1 from the axis
- speed of $m_2 = \omega r_2$, where r_2 is the distance of m_2 from the axis, etc.

  ………………………………………………………………………………

- speed of $m_N = \omega r_N$, where r_N is the distance of m_N from the axis.

  Since the kinetic energy of each point mass is given by $\frac{1}{2}mv^2$, then

- kinetic energy of $m_1 = \frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1\omega^2 r_1^2$
- kinetic energy of $m_2 = \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_2\omega^2 r_2^2$, etc.

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- kinetic energy of $m_N = \frac{1}{2}m_N v_N^2 = \frac{1}{2}m_N\omega^2 r_N^2$

Hence the total kinetic energy $= \frac{1}{2}m_1\omega^2 r_1^2 + m_2\omega^2 r_2^2 + \dots + \frac{1}{2}m_N\omega^2 r_N^2$

$$= \frac{1}{2}[(m_1 r_1^2) + (m_2 r_2^2) + \dots + (m_N r_N^2)]\omega^2$$

$$= \frac{1}{2}I\omega^2 \text{ since } I = [(m_1 r_1^2) + (m_2 r_2^2) + \dots + (m_N r_N^2)]$$

The equation $E_K = \frac{1}{2}I\omega^2$ enables you to calculate how much energy a rotating object 'stores' due to its rotational motion. In addition, it shows that the kinetic energy of a rotating object is proportional to:

- its moment of inertia about the axis of rotation
- the square of its angular speed.

Using flywheels

Flywheels are used in many machines and engines so that the moving parts continue to move when the load on the machine increases and it has to do more work. For example, when a metal press is used to make a shaped object from a sheet of thin metal, the press is able to do the necessary work because the flywheel keeps it moving. Each time a metal sheet is pressed, only a small fraction of the kinetic energy of the flywheel is transferred from the flywheel to the press when the press does work. The flywheel is then speeded up so it has the same amount of kinetic energy before each metal sheet is pressed out.

Flywheels are also used to smooth out the variations of the speed of a motor or an engine when the load varies. For example, when a motorist changes gear, the load on the engine varies during the process. Without a flywheel in the system, the load variation would cause variations in the engine's angular speed – in other words, a jerky ride!

Flywheels are also fitted in some vehicles to store kinetic energy when the vehicle brakes are applied and it slows down. Instead of energy being transferred as heat to the surroundings, some of the vehicle's kinetic energy is transferred to an on-board flywheel to be returned to the vehicle when the accelerator pedal is pressed. In 2009 Formula One motor racing cars started to be fitted with a Kinetic Energy Recovery System (KERS) based on the flywheel principle. The kinetic energy of a flywheel for such use needs to be as large as possible so the flywheel's moment of inertia needs to be as large as practicable. This can be achieved by designing the flywheel so that the outer part near the rim is thicker than the inner part. See Figure 5 in the Summary questions for Topic 1.2.

Links

Kinetic energy, work, and power were looked at in Topic 10.2, Kinetic energy and potential energy, and Topic 10.3, Power, in Year 1 of the *AQA Physics* student book.

Work done

The work done W by a constant torque T when the body is turned through angle θ is given by

$$\boldsymbol{W = T\theta}$$

This can be seen by considering the torque as due to a force F acting at a perpendicular distance d from the axis of rotation. The force acts through a distance $s = \theta d$ when it turns the body through angle θ. Therefore, the work done by the force $W = Fs = Fd\theta = T\theta$ (as $T = Fd$).

Assuming there is no frictional torque, the applied torque $T = I\alpha$.

Therefore, the work done W by the applied torque is given by $W = T\theta = (I\alpha)\theta$.

Since the dynamics equation for rotational motion $\omega_2^2 = \omega_1^2 + 2\alpha\theta$ gives

$\alpha\theta = \frac{1}{2}\omega_2^2 - \frac{1}{2}\omega_1^2$, then

$W = (I\alpha)\theta = I(\frac{1}{2}\omega_2^2 - \frac{1}{2}\omega_1^2) = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$ = the gain of kinetic energy.

Therefore, in the absence of friction, the work done by the torque is equal to the gain of rotational kinetic energy of the body.

Study tip

The number of turns for an angular displacement θ in radians $= \dfrac{\theta}{2\pi}$

Power

For a body rotating at constant angular speed ω, the power P delivered by a torque T acting on the body is given by

$$\boldsymbol{P = T\omega}$$

You can see how this equation arises by considering a constant torque T acting on a body for a time t.

If the body turns through an angle θ in this time, the work done W by the torque is given by

$$W = T\theta$$

Since the power P delivered by the torque is the rate of work done by the torque, then

$$P = \frac{W}{t}$$

Hence $P = \frac{W}{t} = \frac{T\theta}{t} = T\omega$ where the angular speed $\omega = \frac{\theta}{t}$.

If a rotating body is acted on by an applied torque T_1 which is equally opposed by a frictional torque T_F, the resultant torque is zero so its angular speed ω is constant.

- The power P delivered by the applied torque $T = \dfrac{\text{work done } W}{\text{time taken } t} = \dfrac{T\theta}{t} = T\omega$
- The work done per second by the frictional torque $T_F = \dfrac{T_F\theta}{t} = T_F\omega$

In this situation, the rate of transfer of energy due to the applied force is equal to the rate of transfer of energy to the surroundings by the frictional force. So the rotating body does not gain any kinetic energy.

Worked example

A flywheel is rotating at an angular speed of 120 rad s^{-1} on a fixed axle which is mounted on frictionless bearings. The moment of inertia of the flywheel and the axle about the axis of rotation is 0.068 kg m^2.

a Calculate the rotational kinetic energy of the flywheel when it rotates at 120 rad s^{-1}.

b When a braking torque of 1.4 N m is applied to its rim, the flywheel is brought to rest. Calculate the number of turns the flywheel makes as it decelerates to a standstill.

Solution

a $E_K = \frac{1}{2}I\omega^2 = 0.5 \times 0.068 \times 120^2 = 490$ J

b The work done by the braking torque = loss of kinetic energy ΔE_K

Therefore $T\theta = \Delta E_K$ hence $1.4\theta = 490$

Hence $\theta = \frac{490}{1.4} = 350$ rad $= \frac{350}{2\pi} = 56$ turns

Experiment to measure the moment of inertia I of a flywheel

The experiment described below is not part of the specification for the option but it reinforces and brings together many of the ideas you have covered so far. In addition, it provides an opportunity to do some practical work.

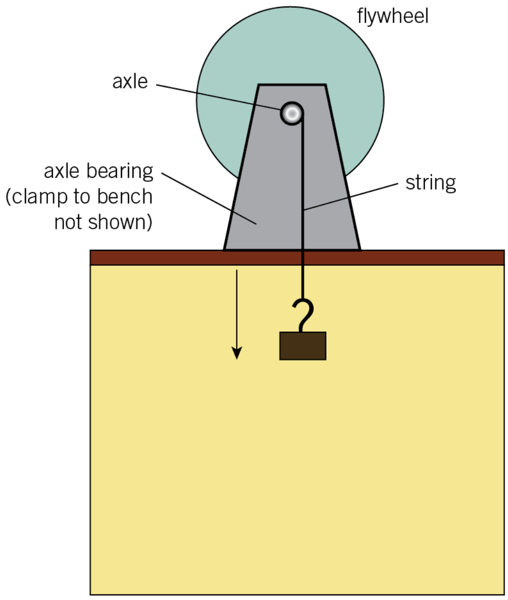

Figure 1 *Measuring the moment of inertia of a flywheel*

An object of known mass M hanging from a string is used to accelerate the flywheel from rest as shown in Figure 1.

The following measurements need to be made:

- the distance, h, fallen by the object from release to when the string unwraps itself from the axle of the flywheel is measured
- the diameter, d, of the axle
- the time taken, t, for the string to unwrap.

The measurements and the known mass M are used in the following calculations:

- The number of turns made by the flywheel as the string unwraps, $N = \dfrac{h}{2\pi d}$
- The maximum angular speed of the flywheel:

  $$\omega = 2 \times \text{the average angular speed} = 2 \times \frac{2\pi N}{t}$$

- The speed of the object at the instant the string unwraps, $v = \dfrac{\omega d}{2}$
- The kinetic energy gained by the flywheel, $\Delta E_{K_F} = \dfrac{1}{2} I\omega^2$, to be calculated in terms of I
- The kinetic energy gained by the object of mass M, $\Delta E_{K_O} = \dfrac{1}{2} Mv^2$

  where $v = \dfrac{\omega d}{2}$
- The potential energy lost by the object of mass M, $\Delta E_P = Mgh$

The moment of inertia of the flywheel can be calculated from the equation below, assuming friction on the flywheel is negligible.

the total gain of kinetic energy, $\Delta E_{K_F} + \Delta E_{K_O}$ = the loss of potential energy, ΔE_P

$$\frac{1}{2} I\omega^2 + \frac{1}{2} \boldsymbol{Mv^2} = \boldsymbol{Mgh}$$

Summary questions

1 A flywheel is rotating at an angular speed of 20 rad s^{-1} on a fixed axle which is mounted on frictionless bearings. The moment of inertia of the flywheel and the axle about the axis of rotation is 0.048 kg m^2.

Calculate:

a the rotational kinetic energy of the flywheel when it rotates at 20 rad s^{-1}

b **i** the torque needed to accelerate the flywheel from rest to an angular speed of 20 rad s^{-1} in 5.0 s

ii the angle which the flywheel turns through in this time while it is being accelerated.

2 A 0.65 kg object hanging from a string is used to accelerate a flywheel on frictionless bearings from rest, as shown in Figure 1. The object falls through a vertical distance of 1.9 m in 4.6 s which is the time the string takes to unwrap from the axle which has a diameter of 8.5 mm. Calculate:

- **a** the potential energy lost by the object in descending 1.9 m
- **b** the kinetic energy of the object 14 s after it was released from rest
- **c** **i** the kinetic energy gained by the flywheel
  - **ii** the moment of inertia of the flywheel.

3 A ball released at the top of a slope rolls down the slope and continues on a flat horizontal surface until it stops. Discuss the energy changes of the ball from the moment it is released to when it stops.

4 A flywheel fitted to a vehicle gains kinetic energy when the vehicle slows down and stops. The kinetic energy of the flywheel is used to make the vehicle start moving again.

- **a** The flywheel is a uniform steel disc of diameter 0.31 m and thickness 0.08 m. Calculate:
  - **i** the mass of the disc
  - **ii** the moment of inertia of the flywheel.

    density of steel = 7800 kg m^{-3}; moment of inertia of a uniform flywheel = $\frac{1}{2}MR^2$

- **b** **i** Calculate the kinetic energy of the flywheel when it is rotating at 3000 revolutions per minute.
  - **ii** The kinetic energy of the flywheel can be converted to kinetic energy of motion of the vehicle in 30 s. Estimate the average power transferred from the flywheel.

1.4 Angular momentum

Learning objectives

→ Define angular momentum and explain why it is important.
→ Explain what is meant by conservation of angular momentum.
→ Define angular impulse.
→ Compare the equations for angular momentum and linear momentum.

Spin at work

Figure 1 *A spinning ice skater*

An ice skater spinning rapidly is a dramatic sight. The skater turns slowly at first, then quite suddenly goes into a rapid spin. This sudden change is brought about by the skater pulling both arms (and possibly a leg!) towards the axis of rotation. In this way the moment of inertia of the skater about the axis is reduced. As a result, the skater spins faster. To slow down, the skater only needs to stretch out his or her arms and maybe a leg. In this way, the moment of inertia is increased. So the skater slows down.

To understand such effects, consider a rotating body with no resultant torque on it. Provided its moment of inertia stays the same, then its angular speed ω does not change. This can be seen by rewriting the equation $T = I\alpha$ where α is the angular acceleration. If the resultant torque T is zero, then the angular acceleration α is zero so the angular speed is constant.

In more general terms, the equation $T = I\alpha$ may be written as:

$$T = \frac{d(I\omega)}{dt}$$

where $\frac{d}{dt}$ is the formal mathematical way of writing 'change per unit time'.

The quantity $I\omega$ is the **angular momentum** of the rotating body. So the above equation tells you that resultant torque T is the rate of change of angular momentum of the rotating object.

Angular momentum of a rotating object $= I\omega$

where I is the moment of inertia of the body about the axis of rotation and ω is its angular speed.

The unit of angular momentum is kg m^2 rad s^{-1} or N m s as explained below in Note 2.

When the resultant torque is zero, then $\frac{d(I\omega)}{dt} = 0$ which means its angular momentum $I\omega$ is constant.

In the ice skater example above, the moment of inertia of the ice skater suddenly decreases when he or she pulls their arms in. Since the angular momentum is constant, the sudden decrease in the moment of inertia causes the angular speed to increase. In specific terms, if the moment of inertia changes from I_1 to I_2 causing the angular speed to change from ω_1 to ω_2 such that

$$I_1\omega_1 = I_2\omega_2$$

Notes

For a rotating object whose moment of inertia I does not change:

1. $T = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\alpha$ because the angular acceleration $\alpha = \frac{d\omega}{dt}$
2. If the object undergoes uniform angular acceleration from rest to reach angular speed ω in time t, the resultant torque acting on it is $T = \frac{I\omega}{t}$. Hence its angular momentum $I\omega = Tt$. This equation shows that the unit of angular momentum can also be given as the unit of torque × the unit of time, i.e., N m s.

Pulsars

Pulsars are rapidly spinning stars. They were first discovered by astronomers in 1967. Regular pulses of radio energy were detected from these stars, which some astronomers called 'LGM' stars. It seemed as if 'little green men' were trying to contact us! That hypothesis was soon abandoned when it was shown that pulsars are in fact rapidly rotating neutron stars which emit radio energy in a beam at an angle to the axis. Each time the beam sweeps round to point towards Earth, radio energy is directed towards you, rather like a light beam from a lighthouse.

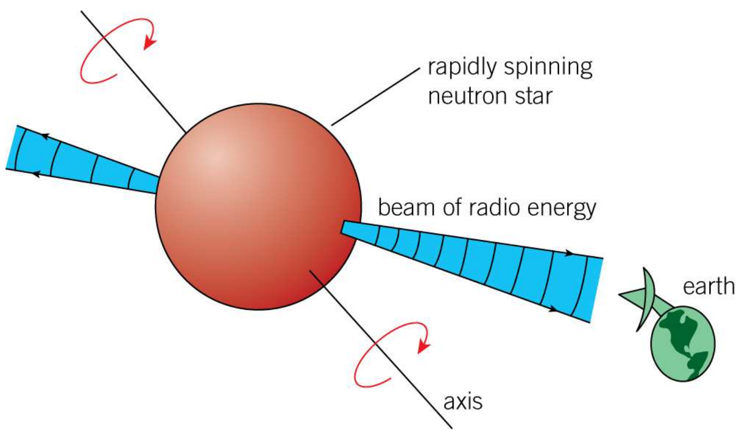

Figure 2 *A pulsar*

Neutron stars are the remnants of large stars. When a massive star runs out of fuel, a huge explosion takes place. The remnants of the explosion are pulled in together by their gravitational attraction, perhaps equivalent to a mass equal to the Sun shrinking to only 15 km or so in diameter. The moment of inertia is therefore made much smaller so the angular speed increases. The pulse frequency from a pulsar is of the order of 1 to 10 Hz, so the rate of rotation is of that order, much, much greater than the Sun's rate which is about once every 25 days!

QUESTION: Estimate the centripetal acceleration at the surface of a pulsar of diameter 10 km spinning at a frequency of 10 Hz.

Angular impulse

When the resultant torque T on a rotating object acts for a short time interval Δt, the angular velocity of the object changes rapidly. This can happen for example when a bat or a racquet hits a moving ball, changing the angular momentum or spin of the ball as well as its direction of motion.

The angular impulse of a short-duration torque is defined as $T\Delta t$. If $\Delta\omega$ is the change of angular momentum due to this torque T, then:

angular impulse = change in angular momentum

which can be written as

$$T\Delta t = I\Delta\omega$$

where I is the moment of inertia of the object about the axis of rotation.

Note

Compare the equation above with the equivalent linear equation $F\Delta t = m\Delta v$, where $F\Delta t$ is the impulse due to a force F acting on an object for a time interval Δt. As explained in Topic 9.1 of the *AQA Physics* student book, the change in momentum of the object, $m\Delta v$, is equal to the impulse $F\Delta t$.

Conservation of angular momentum

In both of the above examples, the ice skater and the pulsar, the angular momentum after the change is equal to the angular momentum before the change because the resultant torque in each case is zero. In other words, the angular momentum is conserved.

Where a system is made up of more than one spinning body, then when two of the bodies interact (e.g., collide), one might lose angular momentum to the other. If the resultant torque on the system is zero, the total amount of angular momentum must stay the same.

Examples of conserving angular momentum

Capture of a spinning satellite

If a spinning satellite is taken on board a space repair laboratory, the whole laboratory is set spinning. The angular momentum of the satellite is transferred to the laboratory when the satellite is taken on board and captured. Unless rocket motors are used to prevent it from turning, then the whole laboratory will spin.

Figure 3 *The Space Shuttle Endeavour capturing a communications satellite*

In this example, assuming the laboratory is not spinning initially:

- the total angular momentum before the satellite is taken on board $= I_1\omega_1$, where I_1 is the moment of inertia of the satellite and ω_1 is its initial angular speed
- the total angular momentum after the satellite has been taken on board and stopped $= (I_1 + I_2)\omega_2$ where ω_2 is the final angular speed and $(I_1 + I_2)$ is the total moment of inertia about the axis of rotation of the satellite as I_2 is the moment of inertia of the space laboratory.

According to the conservation of momentum, $(I_1 + I_2)\omega_2 = I_1\omega_1$

Rearranging this equation gives $I_2\omega_2 = I_1\omega_1 - I_1\omega_2$

This rearranged equation shows that the angular momentum gained by the space laboratory is equal to the angular momentum lost by the satellite.

Object dropped onto a turntable

If a small object is dropped onto a freely-rotating turntable, as shown in Figure 4, so that it sticks to the turntable, the object gains angular momentum and the turntable loses angular momentum.

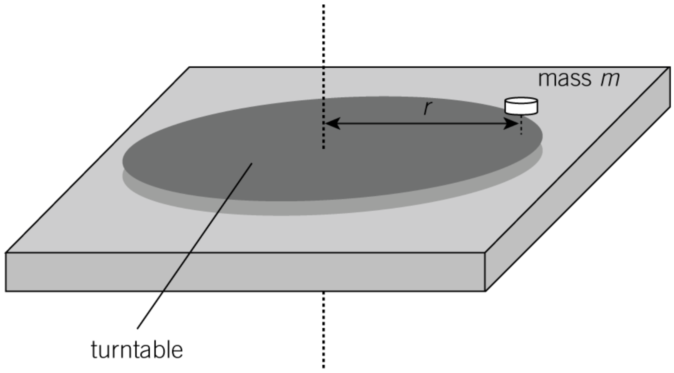

Figure 4 *Measuring the moment of inertia of a freely-rotating turntable*

In this example:

- the total angular momentum of the turntable before the object is dropped onto it $= I_1\omega_1$, where I_1 is the moment of inertia of the turntable and ω_1 is its initial angular speed
- the total angular momentum after the object has been dropped onto the turntable $= (I_1 + I_2)\omega_2$, where ω_2 is the final angular speed and $(I_1 + I_2)$ is the total moment of inertia about the axis of rotation of the turntable.

If the mass of the object is m and its perpendicular distance from the axis of rotation of the turntable is r, then the moment of inertia I_2 of the object about the axis of rotation $= mr^2$.

According to the conservation of momentum, $(I_1 + mr^2)\omega_2 = I_1\omega_1$

Rearranging this equation gives $I_1 = \dfrac{mr^2\omega_2}{\omega_1 - \omega_2}$

Hence I_1 can be found by measuring m, r, ω_1, and ω_2.

Note

The angular momentum of a point mass is defined as its momentum $\times$ its distance from the axis of rotation. For a point mass m rotating at angular speed ω at distance r from the axis, its momentum is $m\omega r$ (as its speed $v = \omega r$) so its angular momentum is $m\omega r^2$.

For a network of point masses m_1, m_2, …, m_N which make up a rigid body,

$$\text{the total angular momentum} = (m_1r_1^{\,2}\omega) + (m_2r_2^{\,2}\omega) + \dots + (m_Nr_N^{\,2}\omega)$$
$$= [(m_1r_1^{\,2}) + (m_2r_2^{\,2}) + \dots + (m_Nr_N^{\,2})]\omega = I\omega$$

Link

Momentum was looked at in Topic 9.1, Momentum and impulse, in Year 1 of the *AQA Physics* student book.

Comparison of linear and rotational motion

When analysing a rotational dynamics situation, it is sometimes useful to compare the situation with an equivalent linear situation. For example, the linear equivalent of a torque T used to change the angular speed of flywheel is a force F used to change the speed of an object moving along a straight line. If the change takes place in time t:

- the change of momentum of the object = Ft
- the change of angular momentum of the flywheel = Tt

Table 1 summarises the comparison between linear and rotational motion.

Table 1 *Comparison between linear and rotational motion*

Linear motion	Rotational motion
displacement s	angular displacement θ
speed and velocity v	angular speed ω
acceleration a	angular acceleration α
mass m	moment of inertia I
momentum mv	angular momentum $I\omega$
force F $F = ma$ $F = \frac{d(mv)}{dt}$	torque $T = Fd$ $T = I\omega$ $T = \frac{d(I\omega)}{dt}$
impulse $F\Delta t = m\Delta v$	angular impulse $T\Delta t = I\Delta\omega$
kinetic energy $= \frac{1}{2}mv^2$	kinetic energy $= \frac{1}{2}I\omega^2$
work done = Fs	work done = $T\theta$
power = Fv	power = $T\omega$

Summary questions

1 A vehicle wheel has a moment of inertia of 5.0×10^{-2} kg m^2 and a radius of 0.30 m.

a Calculate the angular momentum of the wheel when the vehicle is travelling at a speed of 27 m s^{-1}.

b When the brakes are applied, the vehicle speed decreases from 30 m s^{-1} to zero in 9.0 s. Calculate the resultant torque on the wheel during this time.

2 A metal disc X on the end of an axle rotates freely at 240 revolutions per minute. The moment of inertia of the disc and the axle is 0.044 kg m^2.

a Calculate the angular momentum of the disc and the axle.

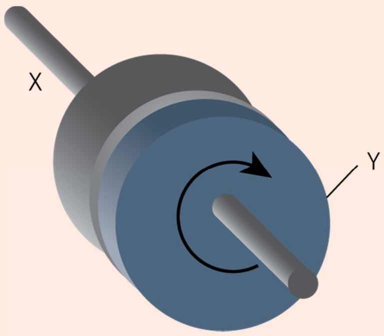

Figure 5

b After a second disc Y that is initially stationary is engaged by X, both discs rotate at 160 revolutions per second. Calculate the moment of inertia of Y.

c Show that the total loss of kinetic energy is 4.6 J.

3 A pulsar is a collapsed star that rotates very rapidly. Explain why a slowly-rotating star that collapses rotates much faster as a result of the collapse.

4 A frictionless turntable is set rotating at a steady angular speed of 20 rad s^{-1}. A small, 0.2 kg, mass is dropped onto the turntable from rest just above it, at a distance of 0.24 m from the centre of the turntable. As a result, the angular speed of the turntable decreases to 18 rad s^{-1}. Calculate the moment of inertia of the turntable about its axis of rotation.