Godalming College

Summer Work: Part 1 – Sequences and Series, Proof and Algebraic Fractions - Solutions

One mark per tick unless stated…

1. $u_1 = 2, u_2 = 5,$
$u_3 = 4 \times 5 - 5 \times 2 = \mathbf{10}$ $u_4 = 4 \times 10 - 5 \times 5 = \mathbf{15}$
$u_5 = 4 \times 15 - 5 \times 10 = \mathbf{10}$ $u_6 = 4 \times 10 - 5 \times 15 = \mathbf{-35}$

✓ for one application of method
✓ for all correct
(2 marks)

2. $a + 2d = 20$ ✓
$a + 6d = 52$ ✓ subtracting gives $4d = 32 \implies d = 8, a = 4$ ✓
$S_n = \frac{10}{2}(8 + 9 \times 8) = 400$ ✓ (4 marks)

3. $3 + (n-1)2 =$ ✓ $136 + (n-1)(-5)$
$3 + 2n - 2 =$ ✓ $136 - 5n + 5$
$7n = 140$ ✓
$n = 20$ (3 marks)

4. Firstly find the sum of all integers from 1 to 300; $a = 1, d = 1, n = 300$ ✓

$$S_{300} = \frac{300}{2}(2 + 1 \times (300 - 1) = 45150$$ ✓

Now find the sum of all multiples of 5 to 300; $a = 5, d = 5, n = \frac{300}{5} = 60$

$$S_{60} = \frac{60}{2}(10 + 5 \times (60 - 1) = 9150$$ ✓

So the sum of all non-multiples of 5 $= 45150 - 9150 = 3600$ ✓ (4 marks)

5.
(a) His 22nd birthday is the third term of the sequence...so $40 + 2 \times 25 = £90$ ✓(1 mark)
(b) Up to and including his 25th birthday =
$S_6 = \frac{6}{2}(2 \times 40 +$ ✓ $25 \times (6-1) = £615$ ✓ (2 marks)
(c) $S_n > 2000 \implies \frac{n}{2}(80 + 25(n-1)) > 2000$ ✓ (multiply both sides by two)
$\implies n(80 + 25n - 25) > 4000$
$\implies 25n^2 + 65n > 4000$ and so $25n^2 + 65n - 4000 > 0$ ✓
From the calculator $n = 11.59, n = -13.79$, ✓ but n must be a positive integer, so $n > 11.59$. So least value of $n = 12$ which corresponds to his 31st birthday. ✓ (4 marks)

6. (a) $S_4 = \frac{a\left(1-\left(-\frac{2}{3}\right)^4\right)}{1--\frac{2}{3}} = 520$ ✓ $\implies a = \frac{520 \times \frac{2}{3}}{1-\left(-\frac{2}{3}\right)^4} = 1080$ as required ✓ (2 marks)

(b) $S_\infty = \frac{1080}{1--\frac{2}{3}}$ ✓ $= 648$ ✓ (2 marks)

(c) $u_{10} - u_9 =$ ✓ $1080\left(\left(-\frac{2}{3}\right)^{10-1} - \left(-\frac{2}{3}\right)^{9-1}\right)$ ✓ $= (-)70.23\ (2d.p.)$ ✓ (3 marks)

7. Sum of first two terms = $a + ar = 34$, ✓
Sum to infinity $\frac{a}{1-r} = 162 \Rightarrow a = 162(1-r)$ ✓
Substituting second equation into the first gives
$162(1-r) + 162r(1-r) = 34$ ✓
$162 - 162r + 162r - 162r^2 = 34$
$162r^2 = 128 \Rightarrow r^2 = \frac{64}{81} \Rightarrow r = \frac{8}{9}$ (as $r > 0$) ✓
$a = 162\left(1 - \frac{8}{9}\right) = 18$ ✓ (5 marks)

8. Sum is greater than 342, so
$S_n = \frac{49(1-(\frac{6}{7})^n)}{1-\frac{6}{7}} > 342 \Longrightarrow 343(1 - \left(\frac{6}{7}\right)^n) > 342$ (divide by 343) ✓
$1 - \left(\frac{6}{7}\right)^n > \frac{342}{343}$ (gather like terms) ✓
$1 - \frac{342}{343} > \left(\frac{6}{7}\right)^n$ (simplifying)
$\frac{1}{343} > \left(\frac{6}{7}\right)^n$ (taking logs of both sides and using rules of logs) ✓
$\log\frac{1}{343} > n\log\frac{6}{7}$ (divide by $\log\frac{6}{7}$ which is negative, so sign reverses) ✓
$n > 37.87$ so least value of $n = 38$ ✓ (5 marks)

9. (a) We know that $\frac{u_{n+1}}{u_n} = \frac{u_{n+2}}{u_{n+1}} = r$, so $\frac{3p-3}{8p} = \frac{2p-10}{3p-3}$ ✓ (multiplying across)
$(3p-3)^2 = 8p(2p-10)$ (multiplying out)
$9p^2 - 18p + 9 = 16p^2 - 80p$ ✓ (gathering like terms on ones side)
$7p^2 - 62p - 9 = 0$ as required ✓ (4 marks)
(b) From calculator, $p = 9$ ✓ (need to say ignore other value as $p > 0$) ✓ (2 marks)
(c) $r = \frac{3p-3}{8p} = \frac{1}{3}$ ✓ ✓ (2 marks)
(d) Also, $a = 8p = 8 \times 9 = 72$ ✓
$S_\infty - S_{10} = \frac{72}{1-\frac{1}{3}} - \frac{72\left(1-\left(\frac{1}{3}\right)^9\right)}{1-\frac{1}{3}} = 108 - 107.994513 = 0.00549$ (3 sig figs) ✓ ✓ (3 marks)

10. Assumption: there are integers p and q for which $12p + 21q = 5$ ✓ (divided by 3)
$4p + 7q = \frac{5}{3}$ but if p is an integer, so is 4p. Likewise 7q is an integer. ✓ The sum of two integers cannot equal a fraction; a contradiction and the assumption is disproved. ✓ (3 marks)

11. Assumption: $\sqrt{3}$ can be written as $\frac{a}{b}$, where a and b are integers with no common factors ✓
So if $\sqrt{3} = \frac{a}{b}$, then $3 = \frac{a^2}{b^2}$ and so $a^2 = 3b^2$ which means that a^2 is a multiple of 3.
But a^2 is a multiple of 3then a is also a multiple of 3 (due to pairing of prime factors) ✓

So $a = 3k$ and so $(3k)^2 = 3b^2$

$9k^2 = 3b^2$ and so $3k^2 = b^2$ ✓

This means that b is also a multiple of 3, which contradicts the fact that a and b are integers with no common factors and so $\sqrt{3}$ is irrational. ✓ (4 marks)

12. Write the following as single fractions in their simplest form

(a) $\frac{3}{x-1} + \frac{2}{x+3} = \frac{3(x+3)+2(x-1)}{(x-1)(x+3)} = \frac{5x+7}{(x-1)(x+3)}$ ✓✓✓

(b) $\frac{6}{x^2-9} - \frac{5}{x^2-x-6} = \frac{6}{(x-3)(x+3)} - \frac{5}{(x-3)(x+2)} = \frac{6(x+2)-5(x+3)}{(x-3)(x+3)(x+2)} = \frac{x-3}{(x-3)(x+3)(x+2)} = \frac{1}{(x+3)(x+2)}$ ✓✓✓

(c) $\frac{3}{x+4} + \frac{2}{x^2+9x+2} = \frac{3}{x+4} + \frac{2}{(x+4)(x+5)} = \frac{3(x+5)+2}{(x+4)(x+5)} = \frac{3x+17}{(x+4)(x+5)}$ ✓✓✓ (9 marks)

13. Write each of the following as partial fractions

(a) $\frac{12}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$ ✓ by the cover-up method, $\frac{12}{(x-1)(x+3)} = \frac{3}{x-1} - \frac{3}{x+3}$ ✓ (2 marks)

(b) $\frac{3}{x^2+5x+6} = \frac{3}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+5}$ ✓ $= \frac{3}{x+2} - \frac{3}{x+3}$ ✓ (2 marks)

(c) $\frac{3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ ✓ $\Rightarrow 3x + 1 = Ax(x-1) + B(x-1) + Cx^2$

Let $x = 1 \Rightarrow C = 4$ ✓

Let $x = 0 \Rightarrow B = -1$ ✓

Equate x^2 terms so $A + C = 0 \Rightarrow A = -4$ giving $\frac{4}{x-1} - \frac{4}{x} - \frac{1}{x^2}$ ✓ (4 marks)

(d) $\frac{2x-5}{(x+2)^2(x-4)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-4}$ ✓

$\Rightarrow 2x - 5 = A(x+2)(x-4) + B(x-4) + C(x+2)^2$

Let $x = 4 \Rightarrow C = \frac{1}{12}$ ✓

Let $x = -2 \Rightarrow B = \frac{3}{2}$ ✓ Equate x^2 terms so $A + C = 0 \Rightarrow A = -\frac{1}{12}$

giving $\frac{3}{2(x+2)^2} + \frac{1}{12(x-4)} - \frac{1}{12(x+2)}$ ✓ (4 marks)

14.

$$
\begin{array}{r|l}
 & 4x \quad + 2 \\
\hline
x^2 + 0x - 4 & 4x^3 + 2x^2 - 3x + 5 \\
 & 4x^3 + 0x^2 - 16x \\
\hline
 & \quad 2x^2 + 13x + 5 \\
 & \quad 2x^2 + 0x - 8 \\
\hline
 & \qquad 13x + 13
\end{array}
$$
✓✓✓

$\frac{13x+1}{(x+2)(x-2)} = \frac{13}{4(x+2)} + \frac{39}{4(x-2)}$ (cover-up method) ✓

So $\frac{4x^3+2x^2-3x+5}{x^2-4} = 4x + 2 + \frac{13}{4(x+2)} + \frac{39}{4(x-2)}$ ✓✓ (6 marks)

15. $\frac{2x+6}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{4}{x+1} - \frac{2}{x+2}$ (cover-up method) ✓

$\frac{4}{x+1} - \frac{2}{x+2} = 4(1+x)^{-1} - 2(2+x)^{-1} = 4(1+x)^{-1} - 2 \times 2^{-1}\left(1+\frac{x}{2}\right)^{-1}$ ✓ ✓

$= 4(1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots$ ✓

$-(1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}(\frac{x}{2})^2 + \frac{(-1)(-2)(-3)}{3!}(\frac{x}{2})^3$ ✓

$= 3 - \frac{7x}{2} + \frac{15x^2}{4} - \frac{31x^3}{8} + \cdots$ ✓ (6 marks)

16. (a) $(5-3x)^{-3} = \frac{1}{125}\left(1-\frac{3}{5}x\right)^{-3}$ ✓

$= \frac{1}{125}[\,1 + (-3)\left(-\frac{3}{5}x\right) + \frac{(-3)(-4)}{2}\left(-\frac{3}{5}x\right)^2 + (\frac{(-3)(-4)(-5)}{6}\left(-\frac{3}{5}x\right)^3 + \cdots]$

$= \frac{1}{125} + \frac{9}{625}x + \frac{54}{3125}x^2 + \frac{54}{3125}x^3 + \cdots$ ✓ ✓ ✓ ✓ (5 marks)

(b) $|x| < \frac{5}{3}$ or $-\frac{5}{3} < x < \frac{5}{3}$ ✓ (1 mark)

17. $\sqrt{4 + (4-k)^2 + 1} = 3$ ✓

$(4-k)^2 = 4$

$4 - k = \pm 2$

$k = 2, 6$ ✓ ✓ (3 marks)

18. $BC = AC - AB = \boldsymbol{i} - 7\boldsymbol{j} - 4\boldsymbol{k}$ ✓

$|BC| = \sqrt{1 + 49 + 16} = \sqrt{66}$ ✓

$\widehat{BC} = \frac{1}{\sqrt{66}}(\boldsymbol{i} - 7\boldsymbol{j} - 4\boldsymbol{k})$ ✓

19. $2q - 3p = 13$ *and* $4q + p = 5$ Solve simultaneously to get $p = -3$ *and* $q = 2$ ✓ ✓

Sub in to get $r = 7$ ✓ (3 marks)

20. D$(2, 5, -1)$

✓ ✓ ✓

Pure Total: 107 marks

Summer Work: Part 2 – Correlation and the Normal Distribution

1. Values should be given to 3 decimal places…

Time (t) hours	1	2	3	4	5	6
Number of bacteria (n)	700	1280	1920	3160	5100	8000
log n	2.845	3.107	3.283	3.500	3.708	3.903

PMCC (from calculator) = 0.99897 showing almost perfect positive correlation between t and log n

So $\log n = mt + c$ and so $n = 10^{mt+c} = 10^c \times 10^{mt} = 10^c \times (10^m)^t = ab^t$, where $a = 10^c$ and $b = 10^m$.
From the equation, $c = 2.66$ and $m = 0.209$, so $a = 10^{2.66} = 457$ and $b = 10^{0.209} = 1.62$ (3sf)
(8 marks)

2. (i) From the calculator, PMCC = 0.6261 (1 mark)

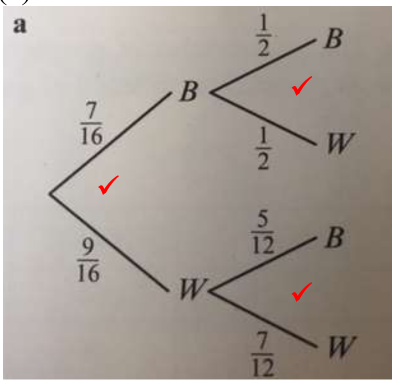
(ii) $H_0: \rho = 0$ and $H_1: \rho > 0$ (one-tailed test)
From tables, p-value for $n = 10$ at 5% level is 0.5494. We see that 0.6261 > 0.5494 and so there is significant correlation – we can reject the null hypothesis and conclude there is positive correlation between the Maths and Physics scores (context). (4 marks)

3. The test is two-tailed as the alternative hypothesis is undirected. From tables, the p-value corresponding to $n = 12$ and significance level of 0.5% at each end is 0.7079, so the results would be significant if r was greater than 0.7079 or less than -0.7079. As $r = 0.-65$ we cannot reject the null hypothesis and she would conclude that there is no evidence of correlation between Maths scores and English scores (context needed). (3 marks)

4. The random variable $X \sim N(30, 5^2)$. Find, to 3 decimal places;
(i) 0.115 (ii) 0.023 (iii)0.497 (iv) $P(28 < X < 32) = 0.311$ (6 marks)

5. (a) $P(X > 280) = 6.68\%$
(b) Using the inverse Normal, drive length would need to be at least 266.8 metres. (4 marks)

6. (a)



(3 marks)

(b) $\left(\frac{7}{12}\times\frac{1}{2}\right)+\left(\frac{9}{16}\times\frac{5}{12}\right)=\frac{29}{64}=0.453\ (3\ s.f.)$ (2 marks)

(c) P(white from A|White from B) = P(white from A ∩ white from B)÷P(white from B)

$$=\left(\frac{9}{16}\times\frac{7}{12}\right)\div\left(1-\frac{29}{64}\right)=\frac{3}{5}$$

(2 marks)

7.

(a) Independent so $P(A)\times P(B)=P(A\cap B)$ so $3P(B)^2=P(A\cap B)$

Using the addition formula $\frac{7}{12}=P(A)+P(B)-P(A\cap B)=4P(B)-3P(B)^2$

Rearrange quadratic to give $3P(B)^2-4P(B)+\frac{7}{12}=0$

Solving gives $P(B)=\frac{7}{6}\ or\ \frac{1}{6}$ so must be $\frac{1}{6}$ (5 marks)

(b) Since A and B are independent, $P(A\cap B)=P(A)\times P(B)=\left(3\times\frac{1}{6}\right)\times\frac{1}{6}=\frac{1}{12}$ (3 marks)

(c) Since A and B are independent $P(B|A')=P(B)=\frac{1}{6}$ (2 marks)

Applied Total: 43 marks