Pure 21 – Numerical Methods

Please **complete** this homework by ___________. Start it early. If you can't do a question you will then have time to ask your teacher for help or go to a drop in session.

Section 1 – Review of previous topics.
Please complete all questions.

1. The functions f(x) and g(x) are given by $f(x) = 2x - 10$ and $g(x) = x^2 - 9, x \epsilon \Re$. Find the value of x for which $f(x) = g(x)$.

2. The graph of $y = ax^2 + bx + c$ has a minimum at (5, -3) and passes through (4, 0). Find the values of a, b and c.

3. Solve the simultaneous equations:

$$x + y = 3$$
$$x^2 - 3y = 1$$

4. Find the set of values for which $12 + 4x > x^2$.

5. Given that $x \neq 3$, find the set of values for which $\frac{5}{x-3} < 2$.

6. Find the possible values of k for the quadratic equation $2kx^2 + 5kx + 5k - 3 = 0$ to have real roots.

7. The point P (2, 1) lies on the graph with equation y = f(x). Write down the coordinates of the point to which P maps under the following transformations:

a) f(x – 4) b) 3f(x) c) 2f(x) – 4

8. The coefficient of x^3 in the binomial expansion of $(3 + bx)^5$ is -720. Find the value of the constant b.

9. In the binomial expansion of $(2k + x)^n$, where k is a constant and n is a positve integer, the coefficient of x^2 is equal to the coefficient of x^3. Prove that $n = 6k + 2$.

10. Write down the first 4 terms of the binomial expansion of $\left(2 + \frac{x}{5}\right)^{10}$. Use this to find an approximate value for 2.1^{10}. Show that the percentage error in your answer is less than 0.1%.

Section 2 – Consolidation of this week's topic.
Please complete all questions.

1) Show that a) $x^3 - 9x + 1 = 0$ has a root between x = 0 and x = 1.
b) $e^{-x} - 9\cos 4x = 0$ has a root in the interval [10, 11] **[4]**

2) Show that $x^3 - 3x + 1 = 0$ can be rearranged to give the iterative formulae:

$$x_{n+1} = \frac{1}{3}(x_n^3 + 1) \quad \text{and} \quad x_{n+1} = \frac{3x_n - 1}{x_n^2}$$

Show that, with a starting value of $x_0 = 0.2$, only one of the iterative formulae is convergent.
Hence, find a root of $x^3 - 3x + 1 = 0$ correct to 7 decimal places. **[10]**

3) $\mathrm{g(x)} = \mathrm{e}^{\mathrm{x}-1} + \mathrm{x} - 6$

a) Show that $\mathrm{g(x)} = 0$ can be written as $\mathrm{x} = \ln(6 - \mathrm{x}) + 1, \quad \mathrm{x} < 6$.
b) The root of $\mathrm{g(x)} = 0$ is α. The iterative formula $\mathrm{x}_{n+1} = \ln(6 - \mathrm{x}_n) + 1$, $\mathrm{x}_0 = 2$ is used to find an approximate value for α. Calculate the values for $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3$ to 4 decimal places .
c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places. **[11]**

4) $\mathrm{f(x)} = 2\sin(\mathrm{x}^2) + \mathrm{x} - 2, \qquad 0 \le \mathrm{x} < 2\pi$

a) Show that $\mathrm{f(x)} = 0$ has a root α between $\mathrm{x} = 0.75$ and $\mathrm{x} = 0.85$.
b) Show that $\mathrm{f(x)}$ can be written as $\mathrm{x} = (\sin^{-1}(1 - 0.5\mathrm{x}))^{\frac{1}{2}}$.
c) Use the iterative formula $\mathrm{x}_{n+1} = (\sin^{-1}(1 - 0.5\mathrm{x}_n))^{\frac{1}{2}}$, $\mathrm{x}_0 = 0.8$ to find the values of $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3$ to 5 decimal places.
d) Show that $\alpha = 0.80157$ is correct to 5 decimal places. **[13]**

5) $f(x) = 2\sqrt{x} + \frac{1}{\sqrt{x}} - 5, \qquad x > 0$

a) Find $f'(x)$

b) The equation $f(x) = 0$ has a root α in the interval [4.5, 5.5]. Using $x_0 = 5$ apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α giving your answer to 2 significant figures.

c) The equation $f(x) = 0$ has another root β in the interval [0.0, 0.5]. Explain why using $x_0 = 0$ or $x_0 = 0.5$ is not suitable as a first approximation.

d) Using $x_0 = 0.1$ apply the Newton-Raphson procedure to $f(x)$ and show that $\beta = 0.048$ to 2 significant figures. **[10]**

6) The Newton-Raphson procedure can be used to find the square root of a number. Suppose we want to find $\sqrt{612}$. We set $x^2 = 612$ and then write $f(x) = x^2 - 612$. We know that $24^2 = 576$ and $25^2 = 625$ so a sensible choice for x_0 is $x_0 = 25$. Use this first approximation to find a second approximation, giving your answer to 2 decimal places. **[2]**

7) Using a similar approach to that in Q6,

a) find a second approximation to the cube root of 612;

b) find a second approximation to the tenth root of 612.
(be careful with your choice of x_0 for part b) **[6]**

8) Sketch on the same axes the graphs of $y = \sqrt{x}$ and $y = e^{-x}$. Use this to state how many solutions there are to the equation $\sqrt{x} - e^{-x} = 0$.

Using $x_0 = 1$ as a first approximation to the solution of $\sqrt{x} - e^{-x} = 0$ apply the Newton-Raphson method to obtain a second approximation giving your answer to 3 decimal places. **[6]**

9) The performance of company shares is predicted to follow the function

$$f(t) = \ln(t+2) + \sin(0.5t) + \sqrt{t}$$

where $f(t)$ represents the value of £1 worth of shares after t years.

An investor purchases £1000 worth of these shares. Using $t_0 = 12$ as a first approximation apply the Newton-Raphson method to find when the investor is predicted to have £7000 worth of shares, giving your answer in years and months. **[5]**

Total: 67 Marks