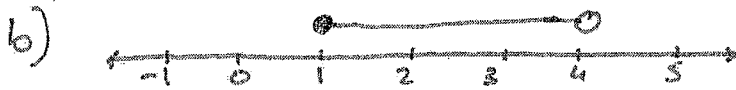
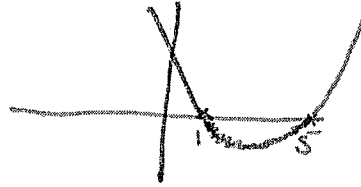


# Pure 14 solutions - section 1

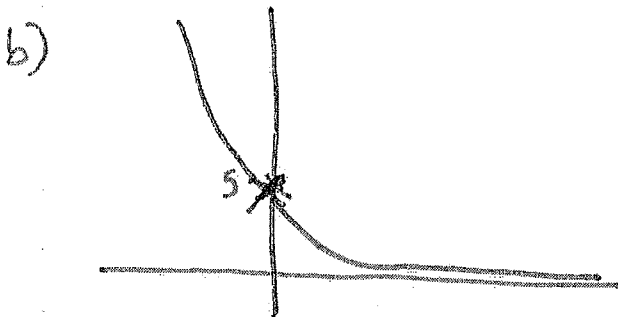
1a) i)  $3x - 5 < 11 - 2x$   
 $4x < 16$   
 $x < 4$

ii)  $x^2 - 6x + 5 \leq 0$   
 $(x - 5)(x - 1) \leq 0$   
critical values: 5, 1  
 $x \quad 1 \leq x \leq 5$



2) a)  $y = ab^x$   
 $(0, 5): 5 = ab^0 \Rightarrow 5 = a$   
 $(2, 1.25): 1.25 = 5b^2$   
 $0.25 = b^2$   
 $0.5 = b$

$$y = 5(0.5)^x$$



3 Prove: if  $n$  is an integer, and  $n^n$  is odd, then  $n$  is odd  
Assume: if  $n$  is an integer and  $n^n$  is odd, then  $n$  is even

if  $n$  is even it can be written as  $2p$

$$\therefore n^n = (2p)^{2p} = (4p^2)^p = 4(4^{p-1}p^{2p}) = \text{even no.}$$

Contradiction: since  $n$  is even  $n^n$  is also even, this contradicts the assumption that  $n^n$  is odd and proves the original statement

$$4a) u_2 = 120, u_5 = 15$$

$$u_n = ar^{n-1}$$

$$\therefore 120 = ar \quad \text{and} \quad 15 = ar^4$$

$$15 = 120r^3$$

$$\frac{1}{8} = r^3$$

$$\frac{1}{2} = r$$

$$b) 120 = a \times \frac{1}{2}$$

$$240 = a$$

$$c) S_{\infty} = \frac{a}{1-r}$$

$$= \frac{240}{1-\frac{1}{2}}$$

$$= 480$$

$$5 \quad x = 5 \sec \theta$$

$$\sec \theta = \frac{x}{5}$$

$$y = 3 \tan \theta$$

$$\tan \theta = \frac{y}{3}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \left(\frac{y}{3}\right)^2 = \left(\frac{x}{5}\right)^2$$

$$\therefore \frac{x^2}{25} - \frac{y^2}{9} = 1$$

$$6) a) V = 2x^2 y = 192 \quad \therefore y = \frac{96}{x^2}$$

$$\text{S.A. } S = (2x^2 + xy + 2xy) \times 2$$

$$S = 4x^2 + 6xy$$

$$S = 4x^2 + 6x \times \frac{96}{x^2}$$

$$S = 4x^2 + \frac{576}{x}$$

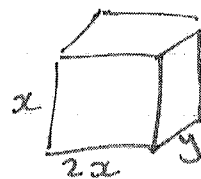
$$b) \frac{dS}{dx} = 8x - 576x^{-2} = 0$$

$$8x = \frac{576}{x^2}$$

$$x^3 = 72$$

$$x = \sqrt[3]{72} = 4.16 \text{ (3 s.f.)}$$

$$\therefore \text{Min } S = 4(x)^2 + \frac{576}{x} = 208 \text{ (3 s.f.)}$$



$$c) \frac{d^2S}{dx^2} = 8 + 1152x^{-3}$$

$$\frac{d^2S}{dx^2} = 8 + 1152(x)^{-3}$$

$$= 24$$

second derivative is positive, therefore point is a minimum

# Pure 19 section 2

1 a  $u = 2x - 1 \therefore x = \frac{1}{2}(u + 1), \frac{du}{dx} = 2$

$$\begin{aligned} \int x(2x-1)^4 dx &= \int \frac{1}{2}(u+1)u^4 \times \frac{1}{2} du \\ &= \frac{1}{4} \int (u^5 + u^4) du \checkmark \\ &= \frac{1}{4} \left( \frac{1}{6}u^6 + \frac{1}{5}u^5 \right) + c \\ &= \frac{1}{4} \left[ \frac{1}{6}(2x-1)^6 + \frac{1}{5}(2x-1)^5 \right] + c \checkmark \\ &= \frac{1}{120} (2x-1)^5 [5(2x-1) + 6] + c \\ &= \frac{1}{120} (10x+1)(2x-1)^5 + c \checkmark \end{aligned}$$

c  $x = \sin u \therefore \frac{dx}{du} = \cos u$

$$\begin{aligned} \int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx &= \int \frac{1}{\cos^3 u} \times \cos u du \checkmark \\ &= \int \sec^2 u du \\ &= \tan u + c \checkmark \\ &= \frac{\sin u}{\cos u} + c \\ &= \frac{x}{\sqrt{1-x^2}} + c \checkmark \end{aligned}$$

e  $u = 2x + 3 \therefore x = \frac{1}{2}u - \frac{3}{2}, \frac{du}{dx} = 2$

$$\begin{aligned} \int (x+1)(2x+3)^3 dx &= \int \left( \frac{1}{2}u - \frac{1}{2} \right) u^3 \times \frac{1}{2} du \\ &= \frac{1}{4} \int (u^4 - u^3) du \\ &= \frac{1}{4} \left( \frac{1}{5}u^5 - \frac{1}{4}u^4 \right) + c \checkmark \\ &= \frac{1}{4} \left[ \frac{1}{5}(2x+3)^5 - \frac{1}{4}(2x+3)^4 \right] + c \checkmark \\ &= \frac{1}{80} (2x+3)^4 [4(2x+3) - 5] + c \\ &= \frac{1}{80} (8x+7)(2x+3)^4 + c \checkmark \end{aligned}$$

b  $u = 1 - x \therefore x = 1 - u, \frac{dx}{du} = -1$

$$\begin{aligned} \int x\sqrt{1-x} dx &= \int (1-u) \times \sqrt{u} \times -1 du \\ &= -\int u^{\frac{1}{2}} - u^{\frac{3}{2}} du \\ &= -\left( \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} \right) + c \checkmark \\ &= -\left( \frac{2}{3}(1-x)^{\frac{3}{2}} - \frac{2}{5}(1-x)^{\frac{5}{2}} \right) + c \checkmark \\ &= -\frac{2}{15} (1-x)^{\frac{3}{2}} (5 - 3(1-x)) + c \\ &= -\frac{2}{15} (2+3x)(1-x)^{\frac{3}{2}} + c \checkmark \end{aligned}$$

d  $x = u^2 \therefore \frac{dx}{du} = 2u$

$$\begin{aligned} \int \frac{1}{\sqrt{x-1}} dx &= \int \frac{1}{u-1} \times 2u du \checkmark \\ &= \int \frac{2(u-1)+2}{u-1} du \\ &= \int \left( 2 + \frac{2}{u-1} \right) du \checkmark \\ &= 2u + 2 \ln|u-1| + c \\ &= 2\sqrt{x} + 2 \ln|\sqrt{x}-1| + c \checkmark \end{aligned}$$

f  $u^2 = x - 2 \therefore x = u^2 + 2, \frac{dx}{du} = 2u$

$$\begin{aligned} \int \frac{x^2}{\sqrt{x-2}} dx &= \int \frac{(u^2+2)^2}{u} \times 2u du \\ &= 2 \int (u^4 + 4u^2 + 4) du \checkmark \\ &= 2 \left( \frac{1}{5}u^5 + \frac{4}{3}u^3 + 4u \right) + c \\ &= 2 \left[ \frac{1}{5}(x-2)^{\frac{5}{2}} + \frac{4}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} \right] + c \checkmark \\ &= \frac{2}{15} (x-2)^{\frac{1}{2}} [3(x-2)^2 + 20(x-2) + 60] + c \checkmark \\ &= \frac{2}{15} (3x^2 + 8x + 32)(x-2)^{\frac{1}{2}} + c \checkmark \end{aligned}$$

$$2 \quad \text{a} \quad x = \sin u \quad \therefore \frac{dx}{du} = \cos u$$

$$x = 0 \Rightarrow u = 0$$

$$x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6}$$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{6}} \frac{1}{\cos u} \times \cos u du \quad \checkmark$$

$$= \int_0^{\frac{\pi}{6}} du \quad \checkmark$$

$$= [u]_0^{\frac{\pi}{6}} \quad \checkmark$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6} \quad \checkmark$$

$$\text{b} \quad u = 2 - x \quad \therefore x = 2 - u, \quad \frac{du}{dx} = -1$$

$$x = 0 \Rightarrow u = 2$$

$$x = 2 \Rightarrow u = 0$$

$$\int_0^2 x(2-x)^3 dx = \int_2^0 (2-u)^3 \times (-1) du \quad \checkmark$$

$$= \int_0^2 (2u^3 - u^4) du \quad \checkmark$$

$$= \left[ \frac{1}{2}u^4 - \frac{1}{5}u^5 \right]_0^2 \quad \checkmark$$

$$= \left( 8 - \frac{32}{5} \right) - (0)$$

$$= \frac{8}{5} \quad \checkmark$$

$$\text{c} \quad x = 2 \sin u \quad \therefore \frac{dx}{du} = 2 \cos u$$

$$x = 0 \Rightarrow u = 0$$

$$x = 1 \Rightarrow u = \frac{\pi}{6}$$

$$\int_0^1 \sqrt{4-x^2} dx$$

$$= \int_0^{\frac{\pi}{6}} 2 \cos u \times 2 \cos u du \quad \checkmark$$

$$= \int_0^{\frac{\pi}{6}} 4 \cos^2 u du$$

$$= \int_0^{\frac{\pi}{6}} (2 + 2 \cos 2u) du \quad \checkmark$$

$$= [2u + \sin 2u]_0^{\frac{\pi}{6}} \quad \checkmark$$

$$= \left( \frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) - (0)$$

$$= \frac{1}{6} (2\pi + 3\sqrt{3}) \quad \checkmark$$

$$\text{d} \quad x = 3 \tan u \quad \therefore \frac{dx}{du} = 3 \sec^2 u$$

$$x = 0 \Rightarrow u = 0$$

$$x = 3 \Rightarrow u = \frac{\pi}{4}$$

$$\int_0^3 \frac{x^2}{x^2+9} dx = \int_0^{\frac{\pi}{4}} \frac{9 \tan^2 u}{9 \sec^2 u} \times 3 \sec^2 u du \quad \checkmark$$

$$= 3 \int_0^{\frac{\pi}{4}} \tan^2 u du$$

$$= 3 \int_0^{\frac{\pi}{4}} (\sec^2 u - 1) du \quad \checkmark$$

$$= 3 [\tan u - u]_0^{\frac{\pi}{4}} \quad \checkmark$$

$$= 3 \left[ \left( 1 - \frac{\pi}{4} \right) - (0) \right]$$

$$= \frac{3}{4} (4 - \pi) \quad \checkmark$$

$$Q3) \quad u^2 = e^{2x} - 2 \rightarrow 2u \frac{du}{dx} = e^{2x} \rightarrow dx = \frac{2u}{e^{2x}} du$$

$$\int_{x=\ln 3}^{x=\ln 4} \frac{e^{4x}}{e^{2x}-2} dx = \int_{u=1}^{u=\sqrt{2}} \frac{e^{4x}}{u^2} \cdot \frac{2u}{e^{2x}} du$$

$$u^2 = 3 - 2 = 1 \quad \checkmark$$

$$u^2 = 4 - 2 = 2 \quad \checkmark$$

$$e^{2x} = 2 + u^2$$

$$e^{3x} = (2 + u^2)^3 \quad \checkmark$$

$$= 8 + 12u^2 + 6u^4 + u^6$$

$$= \int_{u=1}^{u=\sqrt{2}} \frac{2e^{3x}}{u} du \quad \checkmark$$

$$= \int_{1}^{\sqrt{2}} \frac{16 + 24u + 12u^3 + 2u^5}{u} du \quad \checkmark$$

$$= \left[ 16 \ln u + 12u^2 + 3u^4 + \frac{1}{3}u^6 \right]_{1}^{\sqrt{2}} \quad \checkmark$$

$$= (16 \ln \sqrt{2} + 12 \times 2 + 3 \times 4 + \frac{1}{3} \times 8) - (0 + 12 + 3 + \frac{1}{3}) \quad \checkmark$$

$$= 16 \ln \sqrt{2} + 38 \frac{2}{3} - 15 \frac{1}{3}$$

$$= \underline{\underline{\frac{70}{3} + 8 \ln 2}} \quad \checkmark \quad (8)$$

du

$$(Q4) \text{ let } x = \sin u \rightarrow \frac{dx}{du} = \cos u \rightarrow dx = \cos u du$$

$$x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6} \checkmark \rightarrow \int_{\pi/6}^{\pi/3} \sin^2 u \sqrt{1 - \sin^2 u} \times \cos u du$$

$$x = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\pi}{3}$$

$$= \int_{\pi/6}^{\pi/3} \sin^2 u \cos^2 u du \checkmark \quad \begin{array}{l} \sin 2u = 2 \sin u \cos u \\ \left(\frac{1}{2} \sin 2u\right)^2 = (\sin u \cos u)^2 \end{array}$$

$$= \frac{1}{4} \int_{\pi/6}^{\pi/3} \sin^2 2u du \checkmark \quad \sin^2 2u = \frac{1}{2} - \frac{1}{2} \cos 4u$$

$$= \frac{1}{8} \int_{\pi/6}^{\pi/3} 1 - \cos 4u du = \frac{1}{8} \left[ u - \frac{1}{4} \sin 4u \right]_{\pi/6}^{\pi/3} \checkmark$$

$$= \frac{1}{8} \left[ \left( \frac{\pi}{3} + \frac{\sqrt{3}}{8} \right) - \left( \frac{\pi}{6} - \frac{\sqrt{3}}{8} \right) \right] \checkmark$$

(8)

$$= \frac{\pi}{48} + \frac{\sqrt{3}}{32} \checkmark$$

TOTAL  
50 MARKS