Pure 2b Solutions

Section 1.

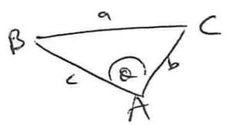
1\. a) $v = \begin{pmatrix} 3 \\ 11 \end{pmatrix} \Rightarrow$ speed $= \sqrt{3^2+11^2} = 11.4\,ms^{-1}$ (3sf)

b) $a = 0,\ u = v \quad s = ut + \frac{1}{2}at^2 \Rightarrow s = 11.4(7) = 79.8\,m$ (3sf)

c) Constant velocity is unrealistic. In reality there would be resistant forces.

2\. a) $\vec{BC} = \vec{AC} - \vec{AB} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$

b)



$a = |\vec{BC}| = \sqrt{8^2+3^2} = \sqrt{73}$

$b = |\vec{AC}| = \sqrt{6^2+3^2} = \sqrt{45}$

$c = |\vec{AB}| = \sqrt{2^2+6^2} = \sqrt{40}$

$$\cos\theta = \frac{b^2+c^2-a^2}{2bc} = \frac{45+40-73}{2\sqrt{45}\sqrt{40}} = \frac{\sqrt{2}}{10}$$

$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{10}\right) = 81.9°$ (3sf)

c) Area $= \frac{1}{2} bc \sin\theta = \frac{1}{2}\sqrt{45}\sqrt{40}\sin(81.9) = 21$

3\. a) F parallel to $\begin{pmatrix} 5 \\ -1 \end{pmatrix} \rightarrow$

$\theta = \tan^{-1}\left(\frac{1}{5}\right) = 11.3°$

i.e. $11.3°$ below horizontal

b) $F = F_1 + F_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3+p \\ -5+q \end{pmatrix}$

$\begin{pmatrix} 3+p \\ -5+q \end{pmatrix} = k\begin{pmatrix} 5 \\ -1 \end{pmatrix} \rightarrow \left.\begin{matrix} 3+p = 5k \Rightarrow k = \frac{3+p}{5} \\ -5+q = -k \Rightarrow k = 5-q \end{matrix}\right\} \frac{3+p}{5} = 5-q$

$\Rightarrow 3+p = 25-5q$

$\Rightarrow 5q + p = 22$

c) $p = 7 \Rightarrow 5q + 7 = 22 \Rightarrow 5q = 15 \Rightarrow q = 3 \Rightarrow F = \begin{pmatrix} 3+7 \\ -5+3 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \end{pmatrix}$

$\Rightarrow$ mag $F = \sqrt{10^2+2^2} = \sqrt{104}$

4\. mag $= \sqrt{2^2+1^2+3^2} = \sqrt{14} \Rightarrow$ unit vector $= \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

5\. $\underline{a} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix},\ \underline{b} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

a) $\underline{a} - \underline{b} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 8 \end{pmatrix}$

b) $2\underline{a} - 3\underline{b} = \begin{pmatrix} 6 \\ -2 \\ 10 \end{pmatrix} - \begin{pmatrix} 12 \\ 0 \\ -9 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ 19 \end{pmatrix}$

c) $\underline{a} - \underline{b}$ is parallel to $\begin{pmatrix} 3 \\ 3 \\ -24 \end{pmatrix}$ because $-3\begin{pmatrix} -1 \\ -1 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -24 \end{pmatrix}$

$2\underline{a} - 3\underline{b}$ is not parallel to $\begin{pmatrix} 3 \\ 3 \\ -24 \end{pmatrix}$ as not multiple of $\begin{pmatrix} -6 \\ -2 \\ 19 \end{pmatrix}$

6\. $\vec{OA} = \begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix},\ \vec{AB} = \begin{pmatrix} 7 \\ -8 \\ -1 \end{pmatrix} \quad C = (2, -2, -1)$

a) $\vec{OB} = \vec{OA} + \vec{AB} = \begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ -8 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix},\quad \vec{OC} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

b) $\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \\ -5 \end{pmatrix}$

c) $|\vec{AC}| = \sqrt{5^2+8^2+5^2} = \sqrt{114}$

d) $|\vec{OC}| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$

7\. $2\underline{a} - \underline{b} = \begin{pmatrix} 3 \\ -6 \\ 10 \end{pmatrix} \Rightarrow 2\begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix} - \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 10 \end{pmatrix} \Rightarrow \begin{pmatrix} 4-p \\ -10-q \\ 12-r \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 10 \end{pmatrix}$

$4 - p = 3 \Rightarrow p = 1$

$-10 - q = -6 \Rightarrow -10 + 6 = q \Rightarrow q = -4$

$12 - r = 10 \Rightarrow r = 2$

8. $\underline{a} = \begin{pmatrix} 3t \\ -12t \\ 4t \end{pmatrix}$ $|\underline{a}| = 39 \Rightarrow \sqrt{(3t)^2 + (12t)^2 + (4t)^2} = 39$

$\Rightarrow 9t^2 + 144t^2 + 16t^2 = 1521$

$\Rightarrow 169t = 1521$

$\Rightarrow t^2 = 9$

$\Rightarrow t = \pm 3$

9 $|\vec{AB}| = \sqrt{2^2+5^2+3^2} = \sqrt{38}$

$\Rightarrow$ Angle with x axis – $\cos^{-1}\left(\frac{-2}{\sqrt{33}}\right) = 108.9°$

y " – $\cos^{-1}\left(\frac{5}{\sqrt{38}}\right) = 35.8°$

" z – $\cos^{-1}\left(\frac{-3}{\sqrt{38}}\right) = 119.1$

Section 2

1 a) $\frac{dx}{dt} = 3t^2$ $\frac{dy}{dt} = 1$ $\therefore \frac{dy}{dx} = \frac{1}{3t^2}$

b) $\frac{dx}{dt} = 3$ $\frac{dy}{dt} = \frac{1}{t^2}$ $\therefore \frac{dy}{dx} = \frac{1}{3t^2}$

c) $\frac{dx}{dt} = -2\sin 2t$ $\frac{dy}{dt} = \cos t$ $\therefore \frac{dy}{dx} = -\frac{\cos t}{2\sin 2t} = -\frac{1}{4}\operatorname{cosec} t$

d) $\frac{dx}{dt} = e^{t+1}$ $\frac{dy}{dt} = 2e^{2t-1}$ $\therefore \frac{dy}{dx} = 2e^{t-2}$

7

2 a) $x = t + \frac{1}{t}$ $\therefore \frac{dx}{dt} = 1 - \frac{1}{t^2}$

$y = t - \frac{1}{t}$ $\therefore \frac{dy}{dt} = 1 + \frac{1}{t^2}$

$\therefore \frac{dy}{dx} = \frac{t^2+1}{t^2-1}$

$t = 3$ $\therefore \frac{dy}{dx} = \frac{10}{8} = \frac{5}{4}$

$x = 3 + \frac{1}{3} = \frac{10}{3}$ $y = 3 - \frac{1}{3} = \frac{8}{3}$

$\therefore$ at P $y - \frac{8}{3} = \frac{5}{4}\left(x - \frac{10}{3}\right)$

$\therefore y = \frac{5x}{4} - \frac{3}{2}$

b) $x^2 - y^2 = \left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2$

$= \left(t^2 + \frac{1}{t^2} + 2\right) - \left(t^2 + \frac{1}{t^2} - 2\right) = 4$

$\therefore x^2 - y^2 = 4$

c) $x^2 - \left(\frac{5x}{4} - \frac{3}{2}\right)^2 = 4$ $\therefore x^2 - \left(\frac{25x^2}{16} + \frac{9}{4} - \frac{15x}{4}\right) = 4$

$\therefore -\frac{9x^2}{16} - \frac{9}{4} + \frac{15x}{4} = 4$

$\therefore 9x^2 - 60x + 100 = 0$

10

discriminant $= (-60)^2 - 4 \times 9 \times 100 = 0$ $\therefore$ tangent touches the curve only once and does not meet the curve again

3 $x = 2t-4, \quad y = \frac{1}{t}$

a) $x=0 \Rightarrow 2t-4=0 \Rightarrow 2t=4 \Rightarrow t=2$

$x=2 \Rightarrow 2t-4=2 \Rightarrow 2t=6 \Rightarrow t=3$ (3)

b) $\int_{x=0}^{x=2} y\,dx = \int_{t=2}^{t=3} \frac{1}{t}(2)\,dt$

$x = 2t-4$

$\Rightarrow \frac{dx}{dt} = 2$

$\Rightarrow dx = 2dt$

$= \int_2^3 \frac{2}{t}\,dt$ (4)

c) $= \left[2\ln t\right]_2^3$

$= 2\ln 3 - 2\ln 2 = 2(\ln 3 - \ln 2) = 2\ln\left(\frac{3}{2}\right) = 0.8109$

(3)

d) $x = 2t-4 \Rightarrow t = \frac{x+4}{2} \Rightarrow y = \frac{1}{\left(\frac{x+4}{2}\right)} = \frac{2}{x+4}$

$\int_{x=0}^{x=2} y\,dx = 2\int_0^2 \frac{1}{x+4}\,dx$ (4)

$= 2\left[\ln(x+4)\right]_0^2 = 2(\ln 6 - \ln 4) = 2\ln\left(\frac{6}{4}\right) = 2\ln\left(\frac{3}{2}\right)$

(14)

4. a) $A \Rightarrow x=0 \Rightarrow 4\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}, \frac{3\pi}{2}$

gives $y = 2\sin\left(\frac{\pi}{2}\right) = 2$ ie A

gives $y = 2\sin\left(\frac{3\pi}{2}\right) = -2$ ie not A.

$B \Rightarrow y=0 \Rightarrow 2\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = \sin^{-1}(0) = 0, \pi$

gives $x = 4\cos(0) = 4$ ie B (2)

gives $x = 4\cos(\pi) = -4$ ie not B

b) $\text{Area} = \int_{\theta \text{ at } A}^{\theta \text{ at } B} y\,dx$

$x = 4\cos\theta$

$\Rightarrow \frac{dx}{d\theta} = -4\sin\theta \Rightarrow dx = -4\sin\theta\,d\theta$

$= \int_{\pi/2}^{0} 2\sin\theta\,(-4\sin\theta)\,d\theta$

$= \int_{\frac{\pi}{2}}^{0} -8\sin^2\theta\,d\theta$

Note: switching limits changes the sign of the integral

$= \int_0^{\pi/2} 8\sin^2\theta\,d\theta$ (4)

c) $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\Rightarrow 8\sin^2\theta = 4(1-\cos 2\theta)$

$\text{Area} = 4\int_0^{\pi/2} (1-\cos 2\theta)\,d\theta$

Area = $4\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$

$= 4\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= 4\left(\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right)$

$= 4\left(\frac{\pi}{2}\right) = 2\pi$ (4)

Area of ellipse = $4 \times$ shaded area = $4(2\pi) = 8\pi$ (10)

5 a) $x = at$, $y = \frac{4a}{t}$, $y = 5a - x$

&

$\frac{4a}{t} = 5a - at$

If $a=0$ $x=0$, $y=0$
therefore $a \neq 0$, so
can $\div$ by a

$\frac{4}{t} = 5 - t$

$4 = 5t - t^2$

$t^2 - 5t + 4 = 0$

$(t-1)(t-4) = 0$

$t=1$, $t=4$

$x=a$, $y=4a$ $\qquad$ $x=4a$, $y=\frac{4a}{4}=a$

ie $S=(a, 4a)$, $T=(4a, a)$ (5)

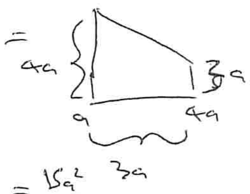
b) $x = at \Rightarrow \frac{dx}{dt} = a$

$\Rightarrow \int y \frac{dx}{dt}\,dt = \int \frac{4a}{t} a\,dt = 4a^2\int \frac{1}{t}\,dt = 4a^2 \ln t + c$ (2)

c) Area between curve & x axis

$= \int_{x=a}^{x=4a} y\,dx = \left[4a^2 \ln t\right]_{t=1}^{t=4} = 4a^2\left[\ln 4 - \ln 1\right] = 4a^2 \ln 4$

Area between line & x axis =



Trapezium = $\frac{1}{2}(3a)(4a+a) = \frac{15a^2}{2}$ (5)

$\Rightarrow$ Required region = $\frac{15a^2}{2} - 4a^2 \ln 4 = \frac{a^2}{2}\left[15 - 8\ln 4\right]$ (12)

(10)

TOTAL = [53]