

1.

Solutions to Pure 22 - ProofSection 1

$$i) a_1 = 2^1 + 1 = 3 \quad a_2 = 2^2 + 1 = 5 \quad a_4 = 2^4 + 1 = 17$$

b) $a_3 = 2^3 + 1 = 9$ 9 is not prime \therefore the sequence doesn't always produce prime numbers.

$$2 \quad a = 2 \quad d = 0.5 \quad u_n = 10 = 2 + (n-1) \times 0.5$$

$$\frac{8}{0.5} + 1 = n = 17$$

$$S_{17} = \frac{17}{2}(2+10) = 102, \quad 102 + (3 \times 10) = 132 \text{ km.}$$

$$3 \quad a) \quad a = 3000 \quad r = 1.04 \quad n = 7 \quad u_7 = 3000 \times 1.04^6 = 3795.96$$

$$b) \quad 6000 = 3000 \times 1.04^n$$

$$2 = 1.04^n \Rightarrow \frac{\log 2}{\log 1.04} = n = 17.67 \dots \text{ so after 18 full years it will have doubled.}$$

$$4. \quad S_\infty = \frac{a}{1-r} = 2a \Rightarrow \frac{1}{1-r} = 2 \Rightarrow 1-r = \frac{1}{2} \Rightarrow r = \frac{1}{2}$$

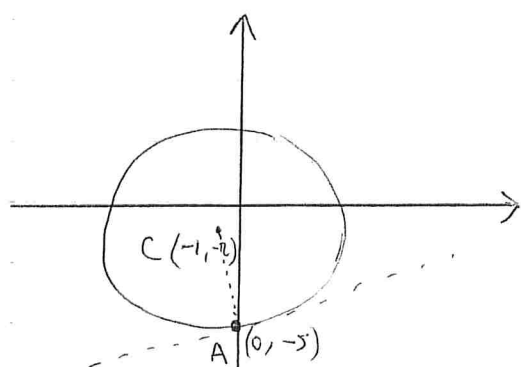
5. $f(3) = 0 \therefore x-3$ is a factor of $f(x)$

$$f(x) = (x-3)(ax^2 + bx + c) \quad a=2 \quad c=2, \quad b=5$$

$$f(x) = (x-3)(2x^2 + 5x + 2) = (x-3)(2x+1)(x+2)$$

$$6. \quad (x+1)^2 - 1 + (y+2)^2 - 4 = 5$$

$$(x+1)^2 + (y+2)^2 = 10$$



$$\text{Gradient of AC} = \frac{-5 - (-2)}{0 - (-1)} = \frac{-3}{1} = -3$$

$$\text{Gradient of tangent} = \frac{1}{3}$$

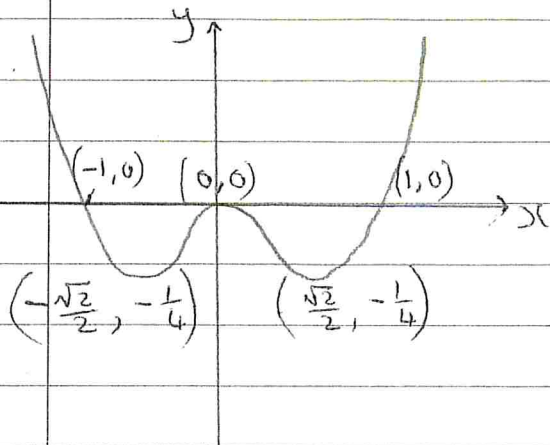
$$y - (-5) = \frac{1}{3}(x - 0)$$

$$\frac{1}{3}x - y = 5 \quad \text{or} \quad y - \frac{1}{3}x = -5$$

$$\text{or} \quad x - 3y = 15$$

2.

$$7) a) y = x^4 - x^2 = x^2(x^2 - 1) = x^2(x+1)(x-1)$$



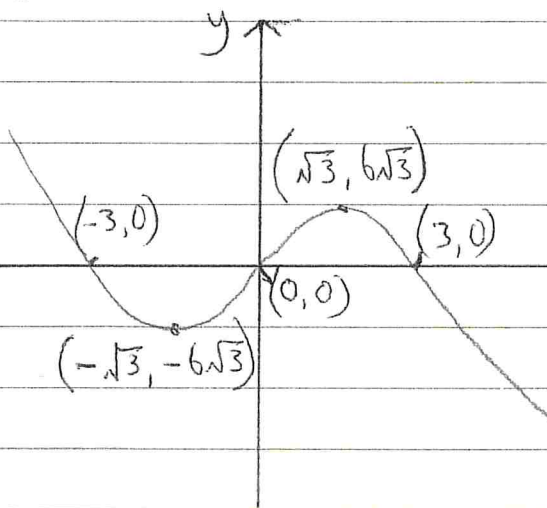
$$y' = 4x^3 - 2x = 0$$

$$2x(2x^2 - 1) = 0$$

$$x = 0, \quad 2x^2 - 1 = 0 \quad x = \pm \frac{\sqrt{2}}{2}$$

$$y = -\frac{1}{4}$$

$$b) y = -x^3 + 9x = x(-x^2 + 9) = x(3-x)(3+x)$$



$$y' = -3x^2 + 9 = 0$$

$$-3x^2 = -9$$

$$x^2 = 3$$

$$x = \pm \sqrt{3}$$

$$y = \pm 6\sqrt{3}$$

Section 2:

1. Assume that M is the largest multiple of 3. $M+3$ is also a multiple of 3 and is larger than M . This contradicts the original assumption. \therefore there is no largest multiple of 3.

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2. Assume there are two odd numbers $2n+1$ and $2m+1$ whose product is even. (n, m integer)

$$(2n+1)(2m+1) = 2nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$$

This is an odd number which contradicts the assumption \therefore the product of two odd numbers is odd.

3. Let $n = 2k+1$, k integer.

$$n^3+1 = (2k+1)^3+1 = 8k^3 + 12k^2 + 6k + 1 + 1 = 8k^3 + 12k^2 + 6k + 2 \\ = 2(4k^3 + 6k^2 + 3k + 1) \text{ which must be even.}$$

\therefore if n is odd then n^3+1 will be even.

4. Assume that $\frac{a}{b}$ is the largest rational number where a and b are integer.

$$\frac{a}{b} + 1 = \frac{a+b}{b} \text{ which is also rational and is larger than } \frac{a}{b}$$

This contradicts the original assumption \therefore there is no largest rational number.

5. Assume that there is some odd number n for which n^2 is even. $n = 2k+1 \Rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

n^2 is odd which contradicts the assumption.

\therefore If n^2 is even then n must be even.

6. Assume that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{a}{b}$ (a, b integer)

Assume also that a and b have no common factors, otherwise $\frac{a}{b}$ could be simplified.

$$\sqrt{2} = \frac{a}{b} \Rightarrow b\sqrt{2} = a \Rightarrow 2b^2 = a^2 \Rightarrow a^2 \text{ is even}$$

$$\Rightarrow a \text{ is even. Let } a = 2n \text{ so } a^2 = 2b^2 \Rightarrow (2n)^2 = 2b^2$$

$$\Rightarrow 4n^2 = 2b^2 \Rightarrow 2n^2 = b^2 \Rightarrow b^2 \text{ is even} \Rightarrow b \text{ is even}$$

If a and b are both even, they have a common factor. (PTO)

4.

This contradicts the original statement \checkmark : $\sqrt{2}$ is irrational.

7. Assume that $1 + \sqrt{2}$ is rational so $1 + \sqrt{2} = \frac{a}{b}$ for some integers a and b . \checkmark

$1 + \sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2} = \frac{a}{b} - 1 \Rightarrow \sqrt{2} = \frac{a-b}{b} \Rightarrow \sqrt{2}$ rational
 $\sqrt{2}$ isn't rational so we have a contradiction. \checkmark
 $\therefore 1 + \sqrt{2}$ is irrational.

8. $25a + 15b = 1 \Rightarrow 5(5a + 3b) = 1 \Rightarrow 5a + 3b = \frac{1}{5}$

This is not possible if a and b are integer \checkmark
 \therefore no integers exist such that $25a + 15b = 1$ \checkmark

9. Assume that there is a finite number of primes say n ,
 $p_1 = 2$ $p_2 = 3$ \dots p_{n-1} and p_n \checkmark

Let $P = p_1 p_2 p_3 \dots p_{n-1} p_n$ (product of all primes) \checkmark

$P+1$ will not be divisible by any number in P therefore $P+1$ is either prime or has prime factors not in P , either of which is a contradiction. Therefore there are infinitely many primes. \checkmark

10. $S_n = a + a+d + a+2d + \dots + a + (n-2)d + a + (n-1)d$

$$S_n = a + (n-1)d + a + (n-2)d + \dots + a + d + a$$

$$2S_n = 2a + (n-1)d + 2a + (n-1)d + \dots + 2a + (n-1)d + 2a + (n-1)d$$

$$2S_n = n [2a + (n-1)d] \Rightarrow S_n = \frac{n}{2} [2a + (n-1)d] \checkmark$$

as required.

11. $S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n \checkmark$$

$$S_n - rS_n = a - ar^n \checkmark$$

$$S_n(1-r) = a(1-r^n) \checkmark$$

$$S_n = \frac{a(1-r^n)}{1-r} \checkmark \text{ as required.}$$